

Schur-Weyl duality

Lecture 24

5/12/2022

Let A, B be algebras over $\mathbb{C}$

$A \otimes B$ as vector spaces can be turned into an algebra:

$$(a \otimes b) \cdot (a' \otimes b') = aa' \otimes bb'$$

Exercise. $\text{Mat}_n(\mathbb{C}) \otimes \text{Mat}_m(\mathbb{C}) = \text{Mat}_{mn}(\mathbb{C})$

Theorem. Let V be an irrep of A ,
 W ———— B .

(i) $V \otimes W$ is an irrep of $A \otimes B$.

(ii) Any irrep M of $A \otimes B$ has the form $M \cong V \otimes W$
for some uniquely determined irreps of A, B .

NB: this is not the same as a rep. $V \otimes W$ of one alg. A .

"Outer product" is sometimes called $V \boxtimes W$.

Proof. (i) $A \rightarrow \text{End}(V)$, $B \rightarrow \text{End}(W)$ are surjective
(density theorem).

$A \otimes B \rightarrow \text{End}(V) \otimes \text{End}(W) \simeq \text{End}(V \otimes W)$
is surjective, so $V \otimes W$ is irred.

(ii) Existence: let A', B' be images of A, B in $\text{End } M$.

$A \otimes B \rightarrow \text{End}(M)$, A', B' fin. dim'l,
 $\begin{matrix} \hookrightarrow & \hookrightarrow \\ A \otimes 1 & 1 \otimes B \end{matrix}$
 $\begin{matrix} \hookrightarrow & \hookrightarrow \\ A' & B' \end{matrix}$
we can suppose that $\dim A, \dim B < \infty$.

Claim: $\text{Rad}(A \otimes B) = (\text{Rad } A) \otimes B + A \otimes \text{Rad } B$.

Proof: Let $J := \text{RHS}$. J is a nilpotent ideal

$$\frac{A \otimes B}{J} \simeq (A/\text{Rad } A) \otimes (B/\text{Rad } B)$$

$J \supset \text{Rad}(A \otimes B)$ $\xrightarrow{\text{semisimple}}$ $\Rightarrow J = \text{Rad}(A \otimes B)$.

$$\frac{A \otimes B}{\text{Rad}(A \otimes B)} = A/\text{Rad } A \otimes B/\text{Rad } B.$$

So M is an irrep of $\frac{A \otimes B}{\text{Rad}(A \otimes B)}$, so
 it is $V \otimes W$, where V, W are reps of $A/\text{Rad } A$ and
 (all of them are $\oplus$ of matrix alg). $\square$ $B/\text{Rad } B$.

Universal enveloping algebra.

Let $\mathfrak{g}$ be a Lie algebra.

Idea: replace $\mathfrak{g}$ with an associative algebra
 $U(\mathfrak{g})$ with 'the same' representations.

Drawback: $U(\mathfrak{g})$ is usually inf. dim'l, even for f.d. $\mathfrak{g}$.

Construction: pick a basis $x_1, \dots, x_n$ in $\mathfrak{g}$.

$$[x_i, x_j] = \sum_{k=1}^n c_{ij}^k x_k, \quad c_{ij}^k \text{ structure constants of } \mathfrak{g}.$$

$U(\mathfrak{g})$ assoc. algebra gen. by $x_1, \dots, x_n$ with rel.

$$x_i x_j - x_j x_i = \sum c_{ij}^k x_k.$$

Basis-free construction of $U(\mathfrak{g})$:

$$\mathfrak{g} \rightarrow T\mathfrak{g} = \mathbb{C} \oplus \mathfrak{g} \oplus (\mathfrak{g} \otimes \mathfrak{g}) \oplus \mathfrak{g}^{\otimes 3} \oplus \dots$$

"noncommutative polynomials in $x_1, \dots, x_n$ "

$T\mathfrak{g} \supset I = (x \otimes y - y \otimes x - [x, y])$ two-sided ideal.

$U(\mathfrak{g}) = T\mathfrak{g}/I$ is a filtered algebra

$U(\mathfrak{g}) = \bigcup U(\mathfrak{g})_k, \quad U(\mathfrak{g})_1 \subset U(\mathfrak{g})_2 \subset \dots, \quad U(\mathfrak{g})_k \cdot U(\mathfrak{g})_l \subset U(\mathfrak{g})_{k+l}.$
elements of degree $\leq k$.

Selecting a basis proves equiv. of these constructions.

If V is a rep. of $\mathfrak{g}$: ($\dim V < \infty$).

$R: \mathfrak{g} \longrightarrow \text{End}(V)$, we have a rep. of
assoc. algebra $U(\mathfrak{g})$:

$$U(\mathfrak{g}) \longrightarrow \text{End}(V)$$

$$R(x_{i_1} \dots x_{i_k}) := R(x_{i_1}) \dots R(x_{i_k}).$$

Vice versa: every rep. of $U(\mathfrak{g})$ as an assoc. alg.
is a rep. of $\mathfrak{g}$ as a Lie algebra.

Take a vector space E

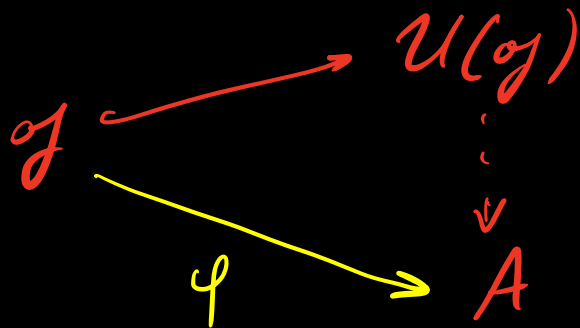
Suppose that $A \subset \text{End}(E)$, subalgebra,

B be a subalgebra in $\text{End}(E)$ s.t. $B \subset \text{End}_A(E)$.
(every element of B commutes with A).

Then E is a rep. of $A \otimes B$.

$$v \in E : (a \otimes b)v = a \cdot b \cdot v = ba v \quad (\text{since } ab = ba \quad \forall a \in A, b \in B)$$

$$\begin{aligned} (a \otimes b) \cdot (a' \otimes b') v &= (a b a' b') v = (a a') (b b') v = \\ &= (a a' \otimes b b') \cdot v. \end{aligned}$$



Let A be an assoc. alg.

$\varphi: g \rightarrow A$ map of vector spaces,

$$\varphi[x, y] = \varphi(x)\varphi(y) - \varphi(y)\varphi(x),$$

~~A is gen. by $\varphi(g)$.~~

$\varphi: g \rightarrow A_L$ is
a Lie alg. homom.

A_L has a Lie bracket $[x, y] := xy - yx$.

Double Centralizer Theorem.

Thm. Let A, B be two subalg. of $\text{End } E$, E f.d.,
s.t. A semisimple, $B = \text{End}_A E$. Then:

(i) $A = \text{End}_B E$.

(ii) B is semisimple;

(iii) As a rep. of $A \otimes B$, E has the decomp.

$$E = \bigoplus V_i \otimes W_i, \quad V_i, W_i \text{ are reps of } A, B \text{ resp.,}$$

All V_i 's are distinct, all W_i 's are distinct.

these are all the reps of A, B .

In particular, this defines a bij. $\text{Irrep}(A) \leftrightarrow \text{Irrep}(B)$.

Proof. A semisimple $\Rightarrow E = \bigoplus_{i \in I} V_i^{d_i}$

$d_i = \dim \operatorname{Hom}(V_i, E)$. If $W_i = \operatorname{Hom}(V_i, E)$,

$E = \bigoplus V_i \otimes W_i$. W_i 's are reps of B ,
irreps or 0.

But W_i cannot be 0 because A acts faithfully on E .

$B = \bigoplus \operatorname{End}(W_i) \leftarrow$ Schur's Lemma.

$A = \bigoplus \operatorname{End}(V_i)$.

Schur-Weyl duality.

V f.d vector space. $n \geq 2$.

$$E = V \otimes \dots \otimes V$$

$$\sigma(v_1 \otimes \dots \otimes v_n) = v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(n)}.$$

E is a rep. of S_n .

Let $\mathfrak{gl}(V)$ be equal to $\text{End}(V)$ with the bracket $[x, y] = xy - yx$.

$$\mathbb{C}S_n \longrightarrow \text{End}(V^{\otimes n}).$$

Let A be the image of $\mathbb{C}S_n$. A semisimple

Thm. The algebra $B = \text{End}_A E$ is the image of $U(\mathfrak{gl}(V))$. B is gen. by the elements

$$\Delta_n(b) = b \otimes 1 \otimes \dots \otimes 1 + 1 \otimes b \otimes \dots \otimes 1 + \dots + 1 \otimes \dots \otimes 1 \otimes b, \quad b \in \mathfrak{gl}(V)$$

Proof. It is clear that $\Delta_n(b) \in B$.

$B = S^n \text{End } V$, need to show: $B \subset \text{Im}(U(\mathfrak{gl}(V)))$.

Lemma. (i) For a f.d. vector space U ,

$S^n U$ is spanned by $u \otimes \dots \otimes u$.

(ii) If A is an algebra, then $S^n A \subset \underbrace{A \otimes \dots \otimes A}_n$ is a subalgebra gen. by $\Delta_n(a)$, $a \in A$.

Proof. (i) $S^n U$ is an irrep of $GL(U)$, so

$\langle GL(U) \cdot u \rangle = S^n U$. Exercise: prove this explicitly.

(ii) Take a ring of sym. poly in n variables:

$p_k(x_1, \dots, x_n) = x_1^k + \dots + x_n^k \leftarrow$ generate $x_1, \dots, x_n$ this ring over $\mathbb{Q}$.

In part., $\exists F: x_1, \dots, x_n = F(p_1, \dots, p_n)$.

Then $a \otimes \dots \otimes a = F(\Delta_n(a), \Delta_n(a^2), \dots, \Delta_n(a^n))$. $\square$

Example: $n=2$: $x_1, x_2 = \frac{1}{2} ((x_1 + x_2)^2 - (x_1^2 + x_2^2))$.

$$x_1, x_2 = \frac{1}{2} (p_1^2 - p_2).$$

$$\begin{aligned} \frac{1}{2} (\Delta_2(a) - \Delta_2(a^2)) &= \frac{1}{2} ((a \otimes 1 + 1 \otimes a)^2 - (a^2 \otimes 1 + \\ &+ 1 \otimes a^2)) = \frac{1}{2} (\cancel{a^2 \otimes 1} + a \otimes a + a \otimes a + \cancel{1 \otimes a^2} - \cancel{a^2 \otimes 1} - \\ &- \cancel{1 \otimes a^2}) = a \otimes a. \end{aligned}$$

Thm. (i) The image A of $\mathbb{C}S_n$ and the image B of $U(\mathfrak{gl}(V))$ are mutual centralizers

(ii) A, B are semisimple. In particular $V^{\otimes n}$ is a semisimple $\mathfrak{gl}(V)$ -module.

(iii) There is a decomp. of $A \otimes B$ -modules

$$V^{\otimes n} = \bigoplus_{\lambda} V_{\lambda} \otimes L_{\lambda}, \quad \lambda \vdash n \text{ partition of } n,$$

V_{λ} are irreps of S_n , L_{λ} are some distinct irreducible $\mathfrak{gl}(V)$ -modules. or 0.

Example: $n = 2$:

$$\begin{array}{ccc} & & V \otimes V \\ & \curvearrowright & \\ S_2 & & \mathfrak{gl}(V) \\ & \longleftarrow & \\ & & V \otimes W \leftrightarrow W \otimes V \end{array}$$

$$V \otimes V = S^2 V \oplus \wedge^2 V$$

$\nearrow$
 S_2 -invar.

$\nwarrow$
 S_2 -antivariant.

$S^2 V, \wedge^2 V$ are
irreps of $\mathfrak{gl}(V)$.

$$V \otimes V \otimes V = (\mathbb{C}_{\text{triv}} \otimes S^3 V) \oplus (\mathbb{C}_{\text{sgn}} \otimes \wedge^3 V) \oplus (\mathbb{C}^2 \otimes S_{2,1})$$

For the group $GL(V)$: $g(v_1 \otimes \dots \otimes v_n) =$
 $= gv_1 \otimes \dots \otimes gv_n.$

$$GL(V) \xrightarrow{\text{diag}} V^{\otimes n} \hookrightarrow \mathbb{C}S_n.$$

Prop. The image of $GL(V)$ in $\text{End}(V^{\otimes n})$
 spans B .

Corollary. As a rep. of $S_n \times GL(V)$,

$$V^{\otimes n} = \bigoplus_{\substack{\text{distinct} \\ \text{irreps}}} V_\lambda \otimes L_\lambda, \quad L_\lambda = \text{Hom}_{S_n}(V_\lambda, V^{\otimes n})$$

or 0.

Fact. (i) L_λ give a complete list of polynomial
reps of $GL(V)$ (or $\mathfrak{gl}(V)$).

(2) Let R be an algebra of polynomials
on $GL(V)$. $R \hookrightarrow GL(V) \times GL(V)$

$$\rho(g, h) \varphi(x) = \varphi(g^{-1} x h).$$

As a rep. of $GL(V) \times GL(V)$,

$$R = \bigoplus_{\lambda} L_{\lambda}^* \otimes L_{\lambda}.$$