

Compact Lie groups & their representations

Lecture 23

1/12/22

Def. A Lie group G (over $\mathbb{R}$) is compact if it is a compact set.

Examples: 1) $SO(2) = S^1 = \{ |z| = 1 \} \subset \mathbb{C}^*$ is compact. Lie groups

2) $T^n = \mathbb{R}^n / \mathbb{Z}^n \simeq S^1 \times \dots \times S^1$ is compact

3) $SO_n = \{ A \in GL_n \mid A^T A = I, \det A = 1 \}$ compact.

$$\sum_{k=1}^n x_{ik} x_{jk} = \delta_{ij} \quad \sum_{k=1}^n x_{ik}^2 = 1. \quad |x_{ik}| \leq 1.$$

3)' O_n compact

4) U_n, SU_n are compact

0) Every finite grp is compact.

$$U_n = \{ A^* A = I \}, \quad A \in \text{Mat}(n, \mathbb{C}), \quad A^* = \overline{A^T}.$$

$$SU_n = U_n \cap SL_n \quad \dim U_n = n^2, \quad \dim SU_n = n^2 - 1.$$

Thm. Every real or complex rep. of a compact Lie group G is completely reducible.

Proof for finite groups: $(\cdot, \cdot)$ inner product on V ,
positive definite nondeg. symmetric bilinear form ($\mathbb{R}$)
/ hermitian sesquilinear form ($\mathbb{C}$)

$$(v, w)_G = \frac{1}{|G|} \sum_{g \in G} (gv, gw). \leftarrow G\text{-invariant inner product}$$

If $U \subset V$ is a subrep. of G , then $U^\perp$ subrep.

$$U^\perp \oplus U = V.$$

Proof for compact groups (for the brave ones!)

$$(v, w)_G = \frac{1}{\mu_G(G)} \int_{g \in G} (gv, gw) d\mu_G$$

Some convex geometry.

$G \curvearrowright E$ E affine space, G acts by affine transforms.

Let $M \subset E$ be a bounded convex set in E ,
 $M \neq \emptyset$.

Convex: $x, y \in M, \Rightarrow [x, y] \subset M$.

$$\text{cent}(M) = \frac{1}{\mu(M)} \int_M x \, \mu(dx). \quad \mu - \text{Lebesgue measure in } E.$$

$$\text{cent}(M) = \frac{1}{n} \sum_{i=1}^n x_i, \quad \text{if } M \text{ finite.}$$

Assume that $\text{Aff}(M) = E$. (M not contained in any proper subspace).

If M convex, $\text{cent}(M) \in M$.

Rem 1. If α - affine transform of E , then
 $\alpha(\text{cent } M) = \text{cent}(\alpha(M))$.

In particular, if $\alpha(M) = M$, $\alpha(\text{cent } M) = \text{cent } M$.

Rem 2. If $\text{Aff } M = E$, then $\text{cent } M \in M^\circ = M \setminus \partial M$.

Proof. Take an affine lin. function f ,
s.t. $f(M) \geq 0$. $\exists m \in M: f(m) > 0$.



$$f(\text{cent } M) = \mu(M)^{-1} \int f(x) \mu(dx) > 0$$

Key Lemma. (fixed point Lemma).

Let G be a compact group of affine transforms of E , and let $M \subset E$ be a nonempty convex set s.t. $gM = M \forall g \in G$. Then $\exists p \in M$:

$$gp = p \quad \forall g \in G.$$

Proof. Take any $x \in M$, let $\text{Conv}(Gx) =: M' \subset M$.
 $\text{cent } M'$ is a G -fixed point. $\square$

Thm. Let G be a compact group, and let V be its real / complex representation. There exists a G -invariant pos. def. (resp hermitian) inner product ^{on V .}

Proof. Take the space of all symm. bilinear forms on V .

The set of pos. def. forms is a convex cone in this space:

$$(\cdot, \cdot)_1, (\cdot, \cdot)_2 > 0 \Rightarrow \lambda(\cdot, \cdot)_1 + \mu(\cdot, \cdot)_2 > 0.$$

G acts on this cone by lin. transf. $\lambda + \mu = 1, \lambda, \mu \geq 0$.

$$\Rightarrow \exists (\cdot, \cdot)_G : (v, w)_G = (gv, gw)_G \quad \forall g \in G. \quad \square$$

Thm Rep. of compact Lie groups are completely reducible.

Pf. Cf. the proof of Maschke's thm. $\square$

Let G be a Lie group, let V be its complex rep.

$$R: G \longrightarrow GL(V).$$

$$dR: \mathfrak{g} \longrightarrow \mathfrak{gl}(V) \simeq \text{End}(V) \text{ rep. of Lie algebra}$$

Thm. Every G -invariant subspace $U \subset V$ is invariant w.r.t. $\mathfrak{g}$. If G connected, the converse is also true.

Proof. 1) Let $U \subset V$ be invariant. Take some $A \in \mathfrak{g}$.

$$\exists g(t) \in G: g(0) = I, g'(0) = A.$$

$$dR(A) = \frac{d}{dt} R(g(t)) \Big|_{t=0}.$$

So for any $u \in U$

$$dR(A)u = \frac{d}{dt} R(g(t))u \Big|_{t=0} \in U.$$

$$2) \quad R(\exp A) = \exp(dR(A)) \quad \forall A \in \mathfrak{g}.$$

$$R(\exp A)u = \sum_{k=0}^{\infty} \frac{1}{k!} dR(A)^k u \in U.$$

U is inv. w.r.t. $\exp(\mathfrak{g})$.

If G connected, $G = \langle \exp(\mathfrak{g}) \rangle$. $\square$.

Corollary. A rep. R of G is irreducible
(completely reducible) $\Leftrightarrow dR$ is irr. (compl. red).
as a rep. of $\mathfrak{g}$.

Relations between complex and real Lie groups.

Def. A connected real subgroup $H \subset G$ for
a conn. complex Lie group G is called its
real form, if $\mathfrak{g} = \mathfrak{f} \oplus i\mathfrak{f}$

Example: $G = SL_n(\mathbb{C})$ $H = SL_n(\mathbb{R})$ is its real form:

$$SL_n(\mathbb{C}) = SL_n(\mathbb{R}) \oplus iSL_n(\mathbb{R}).$$

$$SL_n(\mathbb{C}) = \underset{\text{tr}=0}{\text{Herm}} \oplus \underset{\text{tr}=0}{\text{antiHerm}} = \mathfrak{su}_n \oplus i\mathfrak{su}_n.$$

So $\mathfrak{su}_n$ is a real form for $SL_n(\mathbb{C})$ as well.

Thm. Let $R: G \rightarrow GL(V)$ be a complex rep. of a conn complex Lie group G , and let H be a real form of G .

Then the set of invariant subspaces in V for G and H is the same.

Proof $U \subset V$ is G -invariant $\Leftrightarrow \mathfrak{g}$ -inv.
 H -invariant $\Leftrightarrow \mathfrak{h}$ -inv.

$$dR(g) = dR(f) \oplus idR(f).$$

So U is $dR(g)$ -inv $\Leftrightarrow dR(f)$ -invariant.

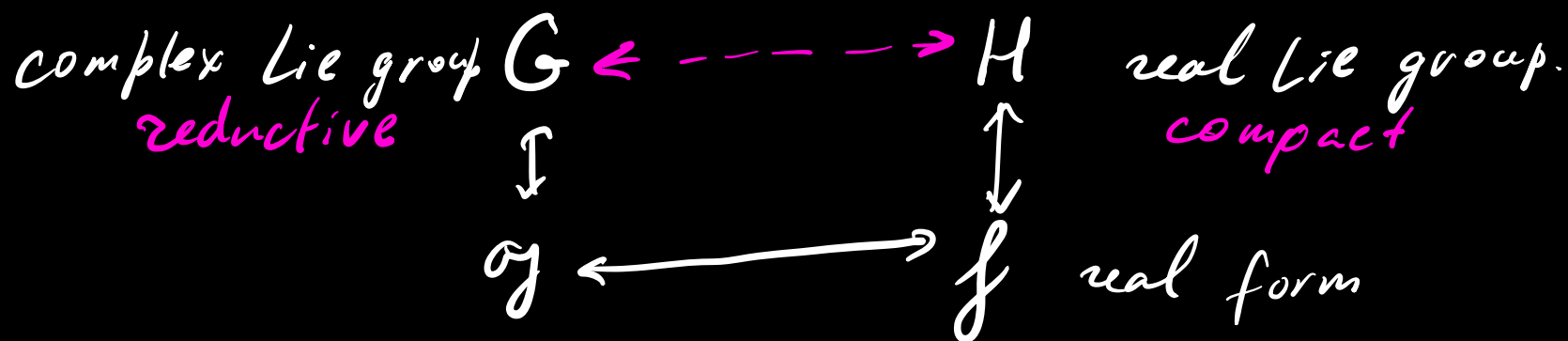
Def. Conn. complex Lie group G is reductive if it has a compact real form.

Examples: $GL_n(\mathbb{C})$, $SL_n(\mathbb{C})$, $(\mathbb{C}^*)^n$, $SO_n(\mathbb{C})$ reductive.

Fact: Every noncomm. simple Lie group is reductive.

Thm. Every f.d. lin. rep. of a reductive group is completely reducible.

Proof. Hermann Weyl's Unitary Trick:



Representations of $SL_2(\mathbb{C})$ and $\mathfrak{sl}_2(\mathbb{C})$.

$$\mathfrak{sl}_2(\mathbb{C}) \ni e, f, h \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

$$[h, e] = 2e \quad [h, f] = -2f \quad [e, f] = h.$$

$$\text{Let } R: SL_2(\mathbb{C}) \rightarrow GL(V)$$

$$dR: \mathfrak{sl}_2(\mathbb{C}) \rightarrow \mathfrak{gl}(V).$$

$$dR(e) = E, \quad dR(f) = F, \quad dR(h) = H.$$

$$[E, F] = H, \quad [H, E] = 2E.$$

Lemma 1 Let v be an eigenvector of H : $Hv = \lambda v$.

Then $E v$ and $F v$ are eigenvectors of H (or $= 0$),

$$H(E v) = (\lambda + 2)(E v) \quad H(F v) = (\lambda - 2)v.$$

Proof. $H E v = E H v + 2 E v = \lambda E v + 2 E v = (\lambda + 2) E v$.

Similarly $H(F v) = (\lambda - 2) F v$. $\square$

Lemma 2. There exists a vector $v_0 \in V$, s.t. $H v_0 = \lambda_0 v_0$

and $E v_0 = 0$. Such a v_0 is called a highest vector of V .

Proof. $v, E v, \dots, E^k v, \dots$ are lin. indep. (eigenvectors for H with different eigenvalues), and V f.d. $\square$

$E v_0 = 0, \quad H v_0 = \lambda_0 v_0 \quad v_0$ highest vector, λ_0 highest weight.

Let $v_1 = F v_0, \quad v_2 = F^2 v_0, \dots, \quad v_k = F^k v_0.$

$$H v_k = (\lambda_0 - 2k) v_k.$$

Lemma 3. $E v_k = c_k v_{k-1}, \quad c_k = k(\lambda_0 - k + 1).$

Proof. Induction upon k . $k=0$: obvious.

$$E v_{k+1} = E F v_k = F E v_k + [E, F] v_k =$$

$$= c_k F v_{k-1} + H v_k = c_k v_k + (\lambda_0 - 2k) v_k = c_{k+1} v_k.$$

$$c_{k+1} = c_k + \lambda_0 - 2k = (k+1)\lambda_0 - k^2 - k = (k+1)(\lambda_0 - k). \quad \square$$

$$\exists n: \quad F^n v_0 = v_n \neq 0, \quad F^{n+1} v_0 = v_{n+1} = 0.$$

$$c_{n+1} = 0, \quad (n+1)(\lambda_0 - n) = 0. \quad \Rightarrow \quad \lambda_0 = n$$

Thm. A highest weight $\lambda_0 \in \mathbb{Z}_{\geq 0}.$

$(v_0, v_1, \dots, v_n) \subset V$ irred. subrep. of V .
 of $\dim n+1$
 with the highest weight n .

$$H v_k = (n - 2k) v_k$$

$$E v_k = k(n - k + 1) v_{k-1}$$

$$F v_k = v_{k+1}.$$

Theorem. Irreps of $sl_2(\mathbb{C})$ are in bij. with $\mathbb{Z}_{\geq 0}$.
 V is uniquely det. by its highest weight n ,
 $\dim V = n + 1$.

Exercise. Let $\mathbb{C}^2 = \langle x, y \rangle$ standard 2-dim'l rep of sl_2 .
 Show that $\mathbb{C}^2 \simeq V_1$, $S^k \mathbb{C}^2 \simeq V_k$.