

Lie groups & the exponential map

Lecture 22

group + smooth manifold in $\text{Mat}(n)$

28/11/22

$$GL(n) \supset G$$

$$T_I G = TG \subset \text{Mat}(n)$$

tangent space to G at $I \in G$.

$$\exp: \text{Mat}(n) \longrightarrow GL(n), \quad \exp X = I + X + \frac{X^2}{2!} + \dots + \frac{X^k}{k!} + \dots$$

Thm 1 If $G \subset GL(n)$ is a Lie group, then

$$\exp TG \subset G.$$

More precisely, $\overset{\cup}{U}(0) \xrightarrow[\text{loc. diffeo}]{} V(I).$

Thm 2 A connected Lie group G is uniquely det.-by TG .

Proof. $TG \supset U(0) \xrightarrow{\sim} V(I) \leftarrow \text{generates } G. \quad \square$

Remark. $\exp$ is, generally speaking, neither surjective nor inj.

a) $\text{Mat}(1, \mathbb{C}) \simeq \mathbb{C} \xrightarrow{\exp} \mathbb{C}^* = GL(1, \mathbb{C})$ is $2\pi i$ -periodic.

b) $TS(2, \mathbb{R}) \longrightarrow SL(2, \mathbb{R})$ is not surjective (Exercise)

Def. Let $G = GL(n, K)$, $H \subset GL(m, K)$ be two Lie groups.
 $f: G \rightarrow H$ is a Lie group homomorphism, if

- it is a group homom;
- it is a smooth map, i.e. the matrix elements are diff. functions.

$$\begin{array}{ccc}
 f: G & \longrightarrow & H \\
 \text{exp}_G \uparrow & & \uparrow \text{exp}_H \\
 d_{\Gamma} f = df: TG & \longrightarrow & TH \quad \text{diff. of a homom.}
 \end{array}$$

Thm. 3 For a homom. $f: G \rightarrow H$, we have

$$f(\text{exp}_G A) = \text{exp}_H df(A) \quad \forall A \in TG.$$

Recall: $g(t) \in GL(n, K)$, $g(0) = I$, $g'(0) = A$, then

$$\text{exp } A = \lim_{n \rightarrow \infty} g\left(\frac{1}{n}\right)^n. \quad e^a = \lim_{n \rightarrow \infty} \left(1 + \frac{a}{n}\right)^n.$$

Proof: Let $g(t) \in G$ be a curve, $g(0) = I$, $g'(0) = A$.
 $f(g(t)) = h(t) \in H$ satisfies $h(0) = I$, $h'(0) = df(A)$

$$f(\exp A) = f\left(\lim_{n \rightarrow \infty} g\left(\frac{1}{n}\right)^n\right) = \lim_{n \rightarrow \infty} f\left(g\left(\frac{1}{n}\right)^n\right) = \lim_{n \rightarrow \infty} h\left(\frac{1}{n}\right)^n = \exp df(A). \quad \square$$

Example: $\det: GL_n(\mathbb{C}) \longrightarrow \mathbb{C}^*$.

Theorem 3 means that $\det(\exp A) = e^{\operatorname{tr} A}$.

Thm 4. A homomorphism $f: G \longrightarrow H$, G connected, is uniquely determined by $df: TG \longrightarrow TH$.

NB: No existence statement so far! (Later...)

Proof: $f: G \longrightarrow H$ homomorphism, G conn.

Knowing df , we can recover $f: V_G(I) \longrightarrow V_H(I)$ for some neighborhoods.

But G is gen. by $V_G(I)$. $\square$.

Exercise: a) $f: G \longrightarrow H$, $\operatorname{Ker} f \subset G$ is a Lie group,
 $T(\operatorname{Ker} f) = \operatorname{Ker}(df)$.

b) If $\operatorname{Im} df = TH$, H connected $\Rightarrow f$ surjective.

The tangent Lie algebra and the adjoint representation.

$$[A, B] = AB - BA \quad A, B \in \text{Mat}(n, K).$$

If A, B nondeg., there is another notation:

$$(A, B) = ABA^{-1}B^{-1} \quad \text{group commutator.}$$

Prop. $[A, B] = \frac{\partial^2}{\partial t \partial s} (\exp(tA), \exp(sB))_{t=s=0}$

Proof: $(\exp(tA), \exp(sB)) = \exp(tA) \cdot \exp(sB) (\exp(tA))^{-1} (\exp(sB))^{-1}$

$$\frac{\partial}{\partial s} (\exp(tA), \exp(sB))_{s=0} = (\exp(tA) B (\exp(tA))^{-1} - B$$

$$\frac{\partial}{\partial t \partial s} (\dots) = A \cdot B - B \cdot A = [A, B]. \quad \square$$

Thm. The tangent space $TG \subset \text{Mat}(n, K)$ is closed w.r.t.
[,]: $A, B \in TG, \quad [A, B] \in TG.$

Proof. $g(s) = (\exp tA, \exp sB) \in G$ for t fixed.

$$g(0) = I, \quad g'(0) = \frac{\partial}{\partial s} (\exp tA \exp sB) \Big|_{s=0} \in TG.$$

$$\frac{\partial^2}{\partial t \partial s} (\exp tA \exp sB) \Big|_{t=s=0} = [A, B] \in TG \quad \square.$$

Def. A subspace $L \subset \text{Mat}(n, K)$ closed w.r.t. $[\cdot, \cdot]$ is called a matrix (linear) Lie algebra.

If $G \subset GL(n, K)$ is a Lie group, then TG is a Lie algebra.

Example 1) $SL_n(K)$: $TSL_n(K) = \{ A \in \text{Mat}_n(K) \mid \text{tr } A = 0 \}.$

$\parallel$
 $\mathfrak{sl}_n(K)$ is a Lie alg.

$$\text{tr } A = \text{tr } B = 0 \Rightarrow \text{tr } [A, B] = 0.$$

2) $TO_n = \mathfrak{o}_n$ skew-sym. matrices. $A^T = -A.$

$$A^T = -A, \quad B^T = -B \quad ([A, B])^T = -[A, B] = [B, A].$$

Properties of $[A, B]$: 1) bilinear;
2) antisymmetric : $[A, B] = -[B, A]$
substitute of associativity: 3) Jacobi identity:

$$[[A, B], C] + [[B, C], A] + [[C, A], B] = 0. \quad \text{Check it!}$$

Another way to view the Jacobi identity:

$$[A, [B, C]] = [[A, B], C] + [B, [A, C]]. \quad \text{Leibniz rule!}$$

Def. A vector space $\mathfrak{g}$ with a binary operation
 $\mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ satisfying (1) - (3) is called
a Lie algebra.

Example: let $\mathfrak{g} \cong \mathbb{R}^3$, $[v, w] = v \times w$ vector product
turns $\mathbb{R}^3$ into a Lie algebra.

Def. A map $\varphi: \mathfrak{g} \rightarrow \mathfrak{h} \leftarrow \text{Gothic h}$
 is a Lie algebra homomorphism, if it is linear
 and $[\varphi(x), \varphi(y)] = \varphi([x, y])$.

Thm. If $f: G \rightarrow H$ is a homom. of Lie groups,
 $df: TG \rightarrow TH$ is a homom. of their
 tangent algebras.

Proof. $A, B \in TG$.

$$\begin{aligned} f(\exp tA, \exp sB) &= (f(\exp tA), f(\exp sB)) = \\ &= (\exp t \cdot df(A), \exp s \cdot df(B)). \end{aligned}$$

$$\begin{aligned} df\left(\frac{\partial}{\partial s}(\exp tA, \exp sB)\right) &= \frac{\partial}{\partial s} \left((\exp t df(A), \exp s df(B)) \right)_{s=0} \\ df[A, B] &= [df(A), df(B)] \quad \square \quad \text{diff. w.r.t. } t \text{ at } t=0. \end{aligned}$$

Def. A linear representation of a Lie group G is a diff. homom. $G \longrightarrow GL(V)$.

$$g \in G \longmapsto a(g): G \longrightarrow G, \quad a(g)x = gxg^{-1}, \quad x \in G.$$

$$d \cdot a(g) =: Ad(g): \mathfrak{g} \longrightarrow \mathfrak{g} \quad \mathfrak{g} = TG.$$

$$\overset{\text{the adjoint operator.}}{Ad(g)} X = g \cdot X g^{-1}$$

$$a(xy) = a(x) \cdot a(y).$$

$$Ad(xy) = Ad(x) \cdot Ad(y).$$

$$Ad: G \longrightarrow GL(\mathfrak{g}) = GL(TG).$$

the adjoint representation.

$$d(Ad) = ad: \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g}). \leftarrow \text{linear maps } \mathfrak{g} \rightarrow \mathfrak{g}.$$

Def $\text{ad}: \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$: adjoint representation of $\mathfrak{g}$.

Thm. $\text{ad}(A) \cdot X = [A, X]$, $A, X \in \mathfrak{g}$.

Proof. $g(t)$ curve in G , $g(0)=I$, $g'(0)=A$.

$$\text{ad}(A) = \left. \frac{d}{dt} \text{Ad}(g(t)) \right|_{t=0}.$$

$$\begin{aligned} \text{ad}(A) \cdot X &= \left. \frac{d}{dt} \text{Ad}(g(t)) \cdot X \right|_{t=0} = \left. \frac{d}{dt} (g(t) X g(t)^{-1}) \right|_{t=0} = \\ &= \left. g'(t) X g(0) - g(0) X g'(t) \right|_{t=0} = AX - XA = [A, X] \quad \square \end{aligned}$$

$$\text{ad}([A, B]) = [\text{ad} A, \text{ad} B]$$

$$[[A, B], C] = [A, [B, C]] + [[A, C], B]. \quad \text{Jacobi identity again!}$$

SU_2 group of unitary matrices with $\det 1$.

$$\left\{ \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \mid a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1 \right\} \leftarrow \text{Exercise.}$$

$\mathfrak{su}_2$ skew-herm. matrices with trace 0.

$$\left\{ X = \begin{pmatrix} ix & y+iz \\ -y+iz & -ix \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\} \quad \det X = x^2 + y^2 + z^2$$

$\det X$ is a pos. def. quadratic form on $\mathfrak{su}_2 \cong \mathbb{R}^3$

$$\det \text{Ad}(g)X = \det g X g^{-1} = \det X$$

$\text{Ad}(g)$ is an orthogonal operator on $\mathfrak{su}_2 \cong \mathbb{R}^3$.

$\text{Ad}: \underset{\substack{\uparrow \\ \text{connected}}}{SU_2} \longrightarrow O_3$ Lie group homomorphism

$$\text{Ad}(SU_2) \subset SO_3.$$

$\text{Ker Ad} = \{\pm I\}$: every matrix commuting with $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ is diagonal.

$\text{Ker ad}: SU_2 \rightarrow SO_3$ is 0.

$SU_2 \rightarrow SO_3$ surjective, because

$SU_2 \rightarrow SO_3$ is surjective.

Then: $\text{Ad}: SU_2 \rightarrow SO_3$ is a surjective map
with $\text{Ker Ad} = \{\pm I\}$.

$$SU_2 / \{\pm I\} \cong SO_3.$$

Topologically: $SU_2 \cong S^3 \xrightarrow{2:1} \mathbb{R}P^3 \cong SO_3.$

$\mathbb{R} \longrightarrow \mathbb{R}^2 / \mathbb{Z}^2$ - two-dim'l torus

$1 \longmapsto (1, \alpha) \quad \alpha \notin \mathbb{Q}.$

$\mathbb{R} \rightarrow \mathbb{R}^2 \rightarrow \mathbb{R}^2 / \mathbb{Z}^2$

