

# Lie Groups

Lecture 21

24/11/22

Lie Group: Group + Smooth manifold

Recall:  $M \subset K^n$  is said to be a  $d$ -dim'l differentiable manifold if  $\forall p \in M \exists p \in U \subset K^n$  <sup>open</sup>  $M \cap U$  is defined (inside  $U$ ) by

$$\begin{cases} f_i(x_1, \dots, x_n) = 0, \\ \dots \end{cases} \quad i = 1, \dots, m, \quad m = n - d,$$

s.t. the Jacobian  $J = \left( \frac{\partial f_i(x)}{\partial x_j} \right)_{x=p}$  has rank  $m$  (at the pt.  $p$ )

$f_i \in C^\infty(K^n)$ .

Two silly examples: 1) Each discrete  $M$  is a 0-dim'l manifold.

Discrete:  $\forall \bar{p} \in M \exists U(p) \subset K^n : U(p) \cap M = \bar{p}$ . ( $x_i = p_i$ )

2) If  $M \subset K^n$  open, it is loc. def. by an empty system of eqns  
← manifold of dim  $n$ .

$J(p) \in \text{Mat}(n \times m)$  of full rank  $m \leq n$

Some minor of order  $m$  is  $\neq 0$ .

W.L.O.G. suppose

$$\begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_m} \end{vmatrix} \neq 0.$$

Implicit function thm  $\Rightarrow \exists \varphi_1, \dots, \varphi_m$  in  $d = n - m$  variables

$$(*) \quad \begin{cases} x_1 = \varphi_1(x_{m+1}, \dots, x_n) \\ \vdots \\ x_m = \varphi_m(x_{m+1}, \dots, x_n) \end{cases}$$

loc. defines  $M$  at  
a neighborhood of  $p \in M$ .

More precisely: if  $p = (p_1, \dots, p_n)$ ,  $\exists (p_1, \dots, p_m) \in \mathcal{U} \subset K^m$   
( $d+m=n$ )  $(p_{m+1}, \dots, p_n) \in V \subset K^d$ : s.t.

$M \cap (\mathcal{U} \times V)$  is the graph of  $\varphi: V \rightarrow \mathcal{U}$  defined by (\*)

Tangent space:  $T_p M$  is formed by the vectors

$(dx_1, \dots, dx_n) \in K^n$  satisfying the system

$$(**) \quad df_i(p) = \sum_{j=1}^n \frac{\partial f_i}{\partial x_j}(p) \cdot dx_j = 0 \quad i=1, \dots, m$$

Space of solutions of  $(**)$  has  $\dim d = n - m$ .

if  $\left( \frac{\partial f_i}{\partial x_j} \right)_{i,j=1}^m \neq 0$ , Free variables:  $dx_{m+1}, \dots, dx_n$ .

$T_p M$  is formed by the velocity vectors of smooth curves passing through  $p$ .

$$T_p M \subset T_p K^n \simeq K^n.$$

Def. A subgroup  $G \subset GL_n(K)$  is said to be a linear Lie group if it is a smooth manifold inside  $\text{Mat}_n(K) \simeq K^{n^2}$ .

$G$  is invariant w.r.t left shifts  $\Leftrightarrow$  the rank cond. can be checked just at one pt.,  $e = g = I$ .

Exercise. For a linear Lie group, show that

$$T_g G = T_e G \cdot g \quad \forall g \in G.$$

$$T_e G = TG \leftarrow \text{not the tangent bundle!}$$

Examples: 1)  $GL_n(K)$  is a linear Lie group:  
group + (open set  $\Rightarrow$  smooth mfd)

$$\dim GL_n(K) = n^2.$$

2)  $SL_n(K) = \{ A \in GL_n(K) \mid \det A = 1 \}$  is an  $(n^2 - 1)$ -dim'l Lie group  
 $\det A = 1$  diff at  $A = I$ :

$$d(\det A) = \text{tr}(dA) = 0.$$

$$TSL_n(K) = \{ X \in \text{Mat}_n(K) \mid \text{tr} X = 0 \}.$$

3)  $O_n(K) = \{ A A^T = I \}$  is a Lie group of dim  $\frac{n(n-1)}{2}$ .

$$\downarrow (A A^T)_I = (dA \cdot A^T + A \cdot (dA)^T)_I = dA + (dA)^T = 0.$$

$$TO_n(K) = \{ X \mid X + X^T = 0 \} \text{ skew-sym matrices.}$$

4)  $U_n \subset L_n(\mathbb{C})$  is def. by  $AA^* = I$ ,  $A^* = (\overline{A^T})$ .

$$\sum_k |x_{ik}|^2 = 1 \quad i=1, \dots, n$$

$$\operatorname{Re} \sum x_{ik} \overline{x_{jk}} = \operatorname{Im} \sum x_{ik} \overline{x_{jk}} = 0$$

$n^2$  real equations  
or  $2n^2$  real param.

$$0 = d(AA^*)_I = dA + dA^*$$

$TU_n = \{X + X^* = 0\}$  - skew-Hermitian matrices

$U_n \subset L_n(\mathbb{C})$  is a real  $n^2$ -dim'l Lie group.

5) Show that  $SU_n = \{AA^* = I, \det A = 1\}$  is a  $n^2 - 1$ -dim'l Lie group.

Prop. Every Lie grp  $G \subset GL_n(K)$  is closed in  $GL_n(K)$ .

Proof.  $G \subset \overline{G} \subset GL_n(K)$ . By continuity,  $\overline{G}$  is subgroup:

$G \subset \overline{G}$  is open.  $g \in \overline{G}$   $gG \subset G$  open in  $\overline{G}$

so  $G \cap gG \neq \emptyset \Rightarrow G = gG \Rightarrow g \in G$ .  $G \subset \overline{G}$  dense.  $\square$

Def.  $M$  is connected if  $M$  cannot be presented as  $M_1 \cup M_2$ ,  $M_1, M_2 \neq \emptyset$ , closed.

If  $X, Y \subset M$  conn.,  $X \cap Y \neq \emptyset$ ,  $X \cup Y$  is connected.

$x \sim y$  if  $x, y \in X \subset M$ ,  $X$  connected.

Classes = conn. components of  $M$ .

Conn. components are open in  $M$ .

and closed:  $X_\alpha = M \setminus \bigcup_{\beta \neq \alpha} X_\beta$  ← other conn. comp.

Def:  $G^\circ$  - the conn. component of  $G$  containing  $I$ .

Prop:  $G^\circ \triangleleft G$  normal subgroup; all conn. components have the form  $gG^\circ$ .

Proof:  $\mu_g: G \rightarrow G$ ,  $h \mapsto gh$  - homeomorphism.

$M_g$  permutes conn. comp.  $g G^\circ = G g$  conn. comp. cont.  $g \in G$ .

In part,  $g \in G^\circ \Rightarrow g G^\circ = G^\circ$  closed under mult.

$\iota: G \rightarrow G \quad h \mapsto h^{-1}$  homeom.  $\Downarrow$

$\iota(G^\circ) = G^\circ \longrightarrow G^\circ$  subgroup.

$g G^\circ = G^\circ g$  normality.  $\square$

Examples.  $SL_n(K)$  is connected.

$\{I + c E_{ij}\}_{\substack{i \neq j \\ c \in K}}$   $E_{ij} = \begin{pmatrix} 0 & 1 \\ & \ddots \\ & & 0 \end{pmatrix}$  connected.

$$SL_n(K) = \langle I + c E_{ij} \rangle = (SL_n(K))^\circ.$$

2) a. Show that  $GL_n(\mathbb{C})$  is connected,

b.  $GL_n(\mathbb{R})^\circ = \{A \mid \det A > 0\}.$

3) Show that  $O_n(\mathbb{R})$  has two conn. comp:

$$O_n^\circ = SO_n(\mathbb{R}) = \{ A \mid \det A = 1, AA^T = I \}.$$

4) Show that  $U_n$  and  $SU_n$  are connected.

Prop. A connected Lie group  $G$  is generated by any neighborhood  $e \in U \subset G$

Proof  $U \subset G$   $\langle U \rangle = \hat{G}$

For any  $g \in G$ ,  $gU \subset g \cdot \hat{G} \Rightarrow g \cdot \hat{G}$  open in  $G$ .  
 $\nwarrow$  ball containing  $G$ .

$G = \bigsqcup g \hat{G}$  connected  $\Rightarrow \hat{G} = G$ .  
 $\uparrow$   
open

The exponential map.

$$X \in \text{Mat}_n(K) \Rightarrow \exp(X) = I + X + \frac{X^2}{2!} + \frac{X^3}{3!} + \dots$$

$$X=0 \Rightarrow \exp X = I. \quad \text{nondeg, convergent.}$$

$$\exp X = I + X + \bar{O}(\|X\|).$$

differential of exp: identity map.

$$\text{Jac}(\exp) \neq 0,$$

$$\exp: \begin{array}{ccc} 0 & \xrightarrow{\quad} & I \\ \cap & & \cap \\ \mathcal{U} & \xrightarrow{\sim} & \mathcal{V} \\ \cap & & \cap \\ \text{Mat}_n(K) & \longrightarrow & \text{GL}_n(K) \end{array} \quad \text{diffeo}$$

$$\text{Loc. inverse: } \log \quad \log(I+X) = \sum_{n \geq 1} (-1)^{n-1} \frac{X^n}{n}.$$

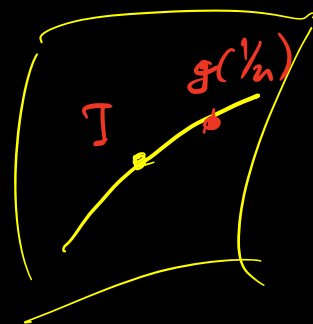
abs. conv. for  $\|X\| < 1$

$$e^x = \lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n.$$

Prop.  $g(t)$ ,  $|t| < c$  smooth curve in  $GL_n(K)$ ,  
Suppose that  $g(0) = I$ ,  $g'(0) = A$ .

Then  $\exp A = \lim_{n \rightarrow \infty} g\left(\frac{1}{n}\right)^n$ .

Proof.  $\frac{d}{dt}(\log g(t))_{t=0} = A$ .



$$\log g(t) = tA + \bar{o}(t)$$

$$g(t) = \exp(tA + \bar{o}(t))$$

$$g\left(\frac{1}{n}\right) = \exp\left(\frac{A}{n} + \bar{o}\left(\frac{1}{n}\right)\right)$$

$$g\left(\frac{1}{n}\right)^n = \exp(A + o(1)) \quad \square.$$

Fact:  $\det(\exp A) = e^{\text{tr} A}$ . In part.,  $\text{tr} A \Rightarrow \det \exp A = 1$ .

Thm. Let  $G \subset GL_n(K)$  be a Lie group.

Then  $\exp: T(G) \rightarrow G$ ,

$\cup$   $\cup$   
 $\exp U(0) \xrightarrow{\sim} V(e)$  diffeo.

Proof.  $A \in T(G) \rightsquigarrow g(t) \in G \quad g(0) = I$

$g'(0) = A$

$g(1/n)^n \in G \quad \forall n \Rightarrow \lim_{n \rightarrow \infty} g(1/n)^n = \exp A \in G.$

$\exp T(G) \rightarrow G$  can be written in loc. coord.  
at  $U(0)$  and  $V(I).$

$G \cap V(I)$  can be def. by  $\left\{ \varphi_i(\overline{x_{m+1}}, \dots, \overline{x_n}) = \overline{x_i} \right.$

$x_{m+1}, \dots, x_n$  - loc. coord.

$T(G) \cap U(0) \quad dx_{m+1}, \dots, dx_n$  - loc. coord.

$d \exp = Id \Rightarrow U(0) \longrightarrow V(I). \quad \square$

Exercises: Check this thm for  $G = SL_n(K)$ ,  
 $O_n(K)$ ,  $U_n$ .

Thm. A conn. Lie grp  $G$  is uniquely  
det. by  $TG$ .

Proof.  $\langle \exp TG \rangle \subset G \Rightarrow \langle \exp TG \rangle = G$ .  
 $\text{Im}(\exp) \supset V(I)$ .