

Irreducibles  $\longrightarrow$  Indecomposables

Lecture 19

Rep.  $V \rightsquigarrow \bigoplus V_i$  indecomposables

17/11/22

Uniqueness?

The Krull-Schmidt theorem.

Thm. Any f.d. representation  $V$  of assoc. algebra  $A$   
can be uniquely decomposed into a direct sum of indec.  
(up to isom. and the order of summands).

Proof. Existence: obvious.

Uniqueness: induction by  $\dim V$ .

$$V = V_1 \oplus \dots \oplus V_m = V_1' \oplus \dots \oplus V_n'$$

$i_s, p_s$  def. similarly.

$$\text{Let } i_t: V_t \hookrightarrow \bigoplus V_j$$

$$p_t: \bigoplus V_j \twoheadrightarrow V_t$$

$$\theta_s = p_i i'_s p'_s i_i : V_i \longrightarrow V_i :$$

$$\theta_1 + \dots + \theta_n = \text{Id}_{V_i}.$$

$$\begin{array}{ccc} & V_i & \\ i_i \downarrow & \uparrow p_i & \\ V_1 \oplus \dots \oplus V_n & \simeq & V'_1 \oplus \dots \oplus V'_n \xrightleftharpoons[p'_s]{i'_s} V'_s \end{array}$$

Lemma. Let  $W$  be a f.d. indec. rep. of  $A$ .  $1_K = \overline{1_K}$

(Fitting Lemma) (i)  $\theta : W \rightarrow W$  is either isom. or nilpotent.  
 (ii) If  $\theta_1, \dots, \theta_n : W \rightarrow W$  are nilpotent,  
 then  $\theta_1 + \dots + \theta_n$  is nilpotent as well.

Proof. (i)

$$\theta = \left( \begin{array}{c|c} \lambda \cdot 1 & \\ \hline & \begin{array}{c} \boxed{\mu \cdot 1} \\ \vdots \\ \boxed{\mu} \end{array} \end{array} \right)$$

gen. eigenspaces  $W_i$

$$\text{So } \theta = \begin{pmatrix} \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda \end{pmatrix}.$$

$$W = \bigoplus W_i \text{ as a rep.}$$

If  $\lambda \neq 0$ ,  $\theta : W \xrightarrow{\sim} W$  isom. Otherwise  $\theta$  nilpotent.

(ii) Ind. by  $n$ .  $n=1$  obvious. Ind. step:  $n-1 \rightarrow n$ .

Assume  $\theta_1, \dots, \theta_n$  nilpotent,  $\theta = \sum \theta_i$  is not. Then  $\theta = \text{isom.}$

$$1 = \theta^{-1} \theta = \sum \theta^{-1} \theta_i, \quad \theta^{-1} \theta_i \text{ are nilpotent.}$$

$$\underbrace{1 - \theta^{-1} \theta_n}_{\text{isomorphism}} = \underbrace{\theta^{-1} \theta_1 + \dots + \theta^{-1} \theta_{n-1}}_{\text{nilpotent by ind. assumption.}}$$

$$(1 - \theta^{-1} \theta_n)^N = 1 + \dots \text{ cannot be nilpotent. } \square.$$

Back to Thm:  $\theta_1 + \dots + \theta_n = \text{Id}_{V_1}$ , so by Lemma some  $\theta_s$  is isomorphism.  
(Let  $\theta_1$  be isom.)

$$\theta_1 = p_1 i_1' p_1' i_1. \quad V_1 \begin{array}{c} \xrightarrow{p_1' i_1'} \\ \xleftarrow{p_1 i_1'} \end{array} V_1'$$

$$V_1' = \text{Im}(p_1' i_1) \oplus \text{Ker}(p_1 i_1'). \text{ So } \text{Ker}(p_1 i_1') = 0,$$

so  $V_1 \xrightarrow[\sim]{p'_1 i_1} V'_1, \quad V'_1 \xrightarrow[\sim]{p_1 i'_1} V_1$  isomorphisms.

$$V = V_1 \oplus \underbrace{V_2 \oplus \dots \oplus V_m}_U = V'_1 \oplus \underbrace{V'_2 \oplus \dots \oplus V'_n}_{U'}$$

We have  $V = V_1 \oplus U = V'_1 \oplus U'$ .

$$U \xhookrightarrow{\quad} V \xrightarrow{\quad} U'. \quad \text{Claim: } h \text{ isomorphism.}$$

$\underbrace{\hspace{10em}}_h$

$$\dim U = \dim U',$$

Enough to show:  $\text{Ker } h = 0$ .

Let  $v \in \text{Ker } h \subset U$ . So  $v \in V_1$ . But  $p_1 v = 0$ .

$p_1 i'_1$  and  $p'_1 i_1$  are isom. Then  $v = 0$ .

So  $V_2 \oplus \dots \oplus V_m \cong V'_2 \oplus \dots \oplus V'_n \Rightarrow V_i \cong V'_i$  after some perm.  $\square$

Exercise. Prove Fitting's Lemma for  $\mathbb{K}$  not alg. closed.

Exercise: Let  $A: \mathbb{R} \rightarrow \mathbb{R}$ , cont. functions

$$f(x) = f(x-1).$$

$$M: \mathbb{R} \rightarrow \mathbb{R}, \quad f(x) = -f(x-1)$$

is an  $A$ -module

Show that •  $A, M$  indep.  $A$ -modules;

•  $A \not\cong M$  as  $A$ -modules, but

•  $A \oplus A \cong M \oplus M$  as  $A$ -modules.

## Extensions of representations.

Problem: given an algebra  $A$ ,  $V, W$  reps of  $A$ ,

Classify reps  $U$  of  $A$  s.t.  $V \subseteq U$  subrep,

$W \cong U/V$  quotient

( $0 \rightarrow V \rightarrow U \rightarrow W \rightarrow 0$  short exact sequence)

Direct sum always exists!  $V \oplus W \cong U$ .

Def.  $U$  is an extension of  $W$  by  $V$ .

Problem: given  $V, W$ , classify extensions of  $W$  with  $V$

Example:  $1 \rightarrow 2 Q = A_2, \quad A = P_Q:$

$$S_1 = (\mathbb{C} \rightarrow 0)$$

$$S_2 = (0 \rightarrow \mathbb{C}) \quad \text{irreducible } A\text{-modules}$$

$$I_{12} = (\mathbb{C} \rightarrow \mathbb{C}) \quad 0 \rightarrow S_2 \rightarrow I_{12} \rightarrow S_1 \rightarrow 0 \quad I_{12}/S_2 \simeq S_1.$$

There are nontriv. ext. of  $S_1$  by  $S_2$ , but  
no ext. of  $S_2$  by  $S_1$ .

Suppose  $U$  is an ext of  $W$  by  $V$ .

As a vector space,  $U \simeq V \oplus W$ ,

$\forall a \in A,$

$$\rho_U(a) = \begin{pmatrix} \rho_V(a) & f(a) \\ 0 & \rho_W(a) \end{pmatrix},$$

$\leftarrow$  not canonically!

$$f(a) \in \text{Hom}(W, V) \\ f: A \rightarrow \text{Hom}_{\mathbb{K}}(W, V).$$

Question: under which conditions on  $f$ , this is a rep.?

$$\rho_u(a)\rho_u(b) = \rho_u(ab)$$

$$\begin{pmatrix} \rho_v(a) & f(a) \\ 0 & \rho_w(a) \end{pmatrix} \cdot \begin{pmatrix} \rho_v(b) & f(b) \\ 0 & \rho_w(b) \end{pmatrix} = \begin{pmatrix} \rho_v(ab) & f(ab) \\ 0 & \rho_w(ab) \end{pmatrix}$$

$$\rho_v(a)f(b) + f(a)\rho_w(b) = f(ab) \quad \text{The cocycle condition.}$$

Such maps are called 1-cocycles of  $A$  with coef.  
in  $\text{Hom}(W, V)$ .

They form a vector space  $Z^1(W, V)$ .

Let  $X: W \rightarrow V$  be a linear map.

$$dX(a) = \rho_V(a) \cdot X - X \cdot \rho_W(a).$$

Prop  $dX$  is a cocycle; it vanishes iff  $X$  is a homomorphism of reps.

Def. Such  $dX(a)$  is called a coboundary.

Proof If  $dX = 0$ ,  $\rho_V(a)X = X\rho_W(a)$ . This means

$$X \in \text{Hom}_A(W, V) \quad \leftarrow \text{homomorphism of reps!}$$

$dX$  is a cocycle: indeed,

$$\begin{aligned} \rho_V(a) dX(b) + dX(a) \cdot \rho_W(b) &= \rho_V(a) (\rho_V(b)X - X\rho_W(b)) \\ &+ (\rho_V(a)X - X\rho_W(a))\rho_W(b) = \rho_V(a)\rho_V(b)X - X\rho_W(a)\rho_W(b) = \\ &= dX(ab). \end{aligned} \quad \square$$

Coboundaries :  $B^1(W, V)$ .

$$\text{Ext}'(W, V) := Z^1(W, V) / B^1(W, V) \quad \text{the 1st extension group.}$$

Prop. (i) Let  $f, f' \in Z^1(W, V)$ ,  $f - f' \in B^1(W, V)$ .

Then the corr. extensions are isomorphic.

Conversely, if  $\varphi: U \xrightarrow{\sim} U'$ ,  $\rho_U(a) = \left( \begin{array}{c|c} * & f(a) \\ \hline * & \end{array} \right)$ ,

$$\varphi(a) = \left( \begin{array}{c|c} 1_V & * \\ \hline 0 & 1_W \end{array} \right). \quad \rho_{U'}(a) = \left( \begin{array}{c|c} * & f(a) \\ \hline * & \end{array} \right)$$

Then  $f - f' \in B^1(W, V)$ .

So  $\text{Ext}'(W, V)$  "classifies" extensions.

(ii) Let  $f \in \text{Ext}'(W, V)$ . Let  $U_f$  be the corr. ext.

$U_f \simeq U_{f'}$  as  $A$ -reps  $\Leftrightarrow f = \lambda f' \quad \lambda \neq 0$ .

The  $\wedge$  isoclasses of ext. are classified by  $\text{PExt}'(W, V)$ ,  
nontriv. ( $\neq V \oplus W$ )

Prop. (i)  $(f - f')(a) = dX(a) = \rho_V(a)X - X \cdot \rho_W(a)$

$$U \longrightarrow U'$$

$$\varphi = \left( \begin{array}{c|c} 1_V & -X \\ \hline 0 & 1_W \end{array} \right).$$

This is an isomorphism

(commutes with  $\rho_U(a), \rho_{U'}(a)$ )

$$\begin{array}{ccc} U & \xrightarrow{\varphi} & U' \\ \rho_U(a) \downarrow & & \downarrow \rho_{U'}(a) \\ U & \xrightarrow{\varphi_a} & U' \end{array}$$

$$\begin{pmatrix} \rho_V(a) & f(a) \\ 0 & \rho_W(a) \end{pmatrix} \begin{pmatrix} 1_V & -X \\ 0 & 1_W \end{pmatrix} = \begin{pmatrix} 1_V & -X \\ 0 & 1_W \end{pmatrix} \begin{pmatrix} \rho_V(a) & f'(a) \\ 0 & \rho_W(a) \end{pmatrix}$$

$$\begin{pmatrix} \rho_v(a) & -\rho_v(a)X + f(a) \\ 0 & \rho_w(a) \end{pmatrix} = \begin{pmatrix} \rho_v(a) & f'(a) - X\rho_w(a) \\ 0 & \rho_w(a) \end{pmatrix}$$

$$-\rho_v(a)X + f(a) = f'(a) - X\rho_w(a).$$

$$f(a) - f'(a) = \rho_v(a)X - X\rho_w(a) = dX(a). \quad \square$$

Exercise. For an acyclic quiver,  $W = S_i$ ,

(possibly  $i = j$ )  $V = S_j$ ,

compute  $\text{Ext}^q(W, V)$ .