

Reps of f.d. associative algebras.

Lecture 18

14/11/2022

- Schur's lemma

- The Density thm: $V = V_1 \oplus \dots \oplus V_r$, $V_i \neq V_j$, irred.

$\oplus \rho_i: A \rightarrow \oplus \text{End } V_i$ is surjective

- Irreps of $\oplus \text{Mat}_{d_i}(k)$: $V_i = k^{d_i}$,
any f.d. rep is a direct sum of V_i 's.

- Any f.d. rep. of an algebra A

admits a finite filtration $0 = V_0 \subset V_1 \subset V_2 \subset \dots \subset V_k = V$,
 V_i/V_{i-1} are irreducible.

Radical of an algebra

(Jacobson radical).

A f.d. algebra over K .

Goal: "measure how non-semisimple an algebra is".

how far from being semisimple?

Def. The radical of A is the set of all $a \in A$
s.t. $\rho_i(a) = 0$ for each irred. rep. $\rho_i: A \rightarrow \text{End } V_i$.

Notation: $\text{Rad}(A)$.

Non-example: if $A = \bigoplus \text{End}(V_i)$, $\text{Rad } A = 0$.

Example: $Q \begin{array}{ccc} \bullet & \xrightarrow{\alpha} & \bullet \\ 1 & & 2 \end{array}$ $A = P_Q = \langle e_1, e_2, e_\alpha \rangle$

Irred: $K \rightarrow 0$
 $0 \rightarrow K$

$\text{Rad } P_Q = \langle e_\alpha \rangle$.

$e_\alpha e_1 = e_2 e_\alpha = e_\alpha$, $e_\alpha^2 = 0$.

$\langle e_\alpha \rangle$ two-sided ideal in P_Q .

Exercise. $\text{Rad}(A)$ is a two-sided ideal.

Thm. (i) Let $I \triangleleft A$ is a two-sided ideal, I nilpotent
 $(I^n = 0 \quad n \gg 0)$. Then $I \subset \text{Rad } A$.

(ii) $\text{Rad}(A)$ is the largest nilpotent two-sided ideal in A .

Proof. (i) Let V be an irrep of A .

$v \in V$ $Iv \subset V$ subrepresentation

If $Iv \neq 0 \Rightarrow Iv = V$

$\exists x \in I: xv = v \quad x^n v = v \neq 0 \quad \forall n \in \mathbb{N}$.

$Iv = 0$, hence $I \subset \text{Rad}(A)$

(ii). Need to show: $\text{Rad}(A)$ is a nilpotent ideal.

regular $A \curvearrowright A$. It admits a filtration

$A_0 \subset A_1 \subset \dots \subset A_n = A$, A_i/A_{i-1} irreducible.

If $x \in \text{Rad}(A)$, x acts on A_i/A_{i-1} by 0.

$\alpha(A_i) \subset A_{i-1}$. So $x_1 \cdots x_n = 0 \quad \forall x_1, \dots, x_n \in A$.
 $\text{Rad}(A)^n = 0$. $\square$

Thm. (i) A f.d. algebra has only fin. many irreps (up to isom)

(ii) These reps $V_1, \dots, V_n$ are fin. dim.

(iii) $A / \text{Rad } A \simeq \bigoplus_i \text{End}(V_i)$

Proof. (ii) V irrep. $v \in V$ $Av \subseteq V$
 $\Rightarrow Av = V, \Rightarrow \dim V < \infty$. $\overset{v}{\underbrace{V}} \leftarrow \text{fin. dim'l, } \neq 0$

(i) Now take $V_1, \dots, V_n$ non-isom. irreps.

$\bigoplus \rho_i : A \longrightarrow \bigoplus_i \text{End } V_i$ is surjective.

$$\dim A \geq \sum_{i=1, \dots, n} (\dim V_i)^2 \geq n$$

So A has only finitely many irreps.

(iii). Let $V_1, \dots, V_n$ be all irreps of A .

$$\rho: A \longrightarrow \bigoplus \text{End } V_i$$

$$\text{Ker } \rho = \text{Rad}(A). \quad \square$$

Corollary. $\dim A \geq \sum_{i=1}^r (\dim V_i)^2$. [analogue of Burnside's formula]

Example: 1. $A = \mathbb{K}[x]/(x^n)$. truncated polynomials
 $\mathbb{K} = \overline{\mathbb{K}}$.

The only irrep of A is $\mathbb{K}$, x acts by 0.

$$\text{Rad}(A) = (x).$$

2. $A = \left\{ \begin{pmatrix} x_{11} & & * \\ & \ddots & \\ 0 & & x_{nn} \end{pmatrix} \right\}$. upper-triang. matrices.

$\cup$
 $I = \left\{ \begin{pmatrix} 0 & * \\ & \ddots \\ 0 & & 0 \end{pmatrix} \right\}$ is nilpotent.

$A \curvearrowright \mathbb{K}$, $Av = x_{ii}v$ are irreps.

$$\dim A = n = \dim I. \quad I = \text{Rad}(A)$$

Several definitions of a semisimple algebra.

Def. (recall) A rep. V is semisimple $\Leftrightarrow V = \bigoplus V_i$, V_i irred.

Def-thm. A is semisimple if any of these equivalent properties holds:

(1) $\text{Rad } A = 0$

(2) $\dim A = \sum (\dim V_i)^2$ V_i all irred. reps.

(3) $A \simeq \bigoplus \text{Mat}_{d_i}(K)$

(4) Any f.d. rep of A is semisimple

(5) The regular rep. of A is semisimple.

Proof. (1) $\Leftrightarrow$ (2) $\dim A - \dim \text{Rad}(A) = \sum (\dim V_i)^2$.

If $\text{Rad}(A) = 0$, $A \simeq \bigoplus \text{Mat}_{d_i}(K)$ (1) $\Rightarrow$ (3).

If $A = \bigoplus \text{Mat}_{d_i}(K) \Rightarrow \text{Rad}(A) = 0$ (3) $\Rightarrow$ (1)

(Last time): (3) $\Rightarrow$ (4) (4) $\Rightarrow$ (5) obvious.

Let us prove that (5) $\Rightarrow$ (3). Let $A = \bigoplus n_i V_i$,

$\text{End}_A(A) = ?$ · Schur's Lemma: no V_i can be mapped into distinct V_j .

$$\text{End}_A(V_i) = K. \quad \text{End}_A(n_i V_i) \simeq \text{Mat}_{n_i}(K).$$

$$\text{End}_A(A) \simeq \bigoplus \text{Mat}_{n_i}(K).$$

But $\text{End}_A A \simeq A^{\text{op}}$.

$$A^{\text{op}} \simeq \bigoplus \text{Mat}_{n_i}(K).$$

$$A \simeq \left(\bigoplus \text{Mat}_{n_i}(K) \right)^{\text{op}} \simeq \bigoplus \text{Mat}_{n_i}(K).$$

$\text{End}_A A \simeq A^{\text{op}}$:

$\varphi: A \rightarrow A$.

Endomorphism of a rep

$$\begin{aligned} \varphi(1) &= x \\ \psi(1) &= y \end{aligned}$$

$$\varphi \circ \psi(a) = a y x$$

$$\begin{aligned} \varphi(1) &= x \\ \varphi(a) &= ax \end{aligned}$$

$$\varphi \leftrightarrow x.$$

Characters of reps

A algebra, V f.d. rep $\rho: A \rightarrow \text{End } V$.

$$\chi_V(a) = \text{tr}_V \rho(a).$$

$\chi_V: A \rightarrow \mathbb{K}$ linear function.

Linear subspace
↓

$$[A, A] := \langle xy - yx \mid x, y \in A \rangle. \quad [A, A] \subset \text{Ker } \chi_V.$$

$$\chi_V: A/[A, A] \rightarrow \mathbb{K}.$$

Exercise: If $V \supset U$ subrep, then $\chi_V = \chi_U + \chi_{V/U}$.

Thm. (i) $V_1, \dots, V_r$ irreps of A . χ_{V_i} are lin. indep.

(ii) If A is fin. dim'l semisimple algebra,
 χ_V 's form a basis of $(A/[A, A])^*$.

Proof. (i) If $V_1, \dots, V_r$ nonisom. irreps of A

$\bigoplus \rho_i: A \rightarrow \bigoplus \text{End } V_i$ is surjective (density thm).

If $\sum \lambda_i \chi_{v_i}(a) = 0$ for all a ,
 then $\sum \lambda_i \text{Tr}(M_i) = 0 \quad \forall M_i \in \text{End } V_i \Rightarrow \lambda_1 = \dots = \lambda_r = 0$.

(ii) Lemma. $[\text{Mat}_d(K), \text{Mat}_d(K)] = \mathfrak{sl}_d(K) = \{X \in \text{Mat}(d, K) \mid \text{Tr} X = 0\}$.

Proof: $\subset$ obvious ($\text{tr}(XY - YX) = 0$).

Need to show: LHS $\supset E_{ij} = \begin{pmatrix} 0 & \dots & 1 & \dots & 0 \\ & & & & \\ & & & & \\ & & & & \\ 0 & 0 & 0 & & 0 \end{pmatrix}, \quad i \neq j$

$$\hookrightarrow E_{ii} - E_{jj} = \begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & -1 & & \\ & & & \ddots & \\ 0 & & & & 0 \end{pmatrix}$$

$$E_{ij} = [E_{im}, E_{mj}] \quad i \neq j$$

$$[E_{ij}, E_{ji}] = E_{ii} - E_{jj} \quad \square$$

Proof of (ii) If $A = \bigoplus_{i=1}^r \text{Mat}_{d_i}(K)$.

$$[A, A] = \bigoplus_{i=1}^r \mathfrak{sl}_{d_i}(K)$$

$A/[A, A] \simeq K^r$. So χ_{v_i} form a basis.

$\nearrow$ this algebra has r irred. reps

Thm. (Jordan-Hölder). Let V be a fin. dim'l rep of A .

$$0 = V_0 \subset V_1 \subset \dots \subset V_n = V, \quad 0 = V'_0 \subset V'_1 \subset \dots \subset V'_m = V,$$

$$W_i = V_i / V_{i-1}, \quad W'_i = V'_i / V'_{i-1} \quad \text{are irred.}$$

Then $n = m$, $W_i \cong W'_{\sigma(i)}$ for some perm. $\sigma \in S_n$.

Proof For $\text{char } K = 0$.

$$\chi_V = \sum \chi_{W_i} = \sum \chi_{W'_i}. \quad \text{But } \chi_W, \chi_{W'} \text{ are lin. indep.} \Rightarrow$$

$$\Rightarrow W_i \cong W'_{\sigma(i)}. \quad \square$$

Rem. Does not work in $\text{char } K = p$: $\text{mult } W \equiv \text{mult } W' \pmod{p}$.

Fact. Jordan-Hölder thm holds in arbitrary char.
(see Etingof's book, § 3.7).