

Representations of (associative) algebras Lecture 17

A associative algebra, with 1 . $\mathbb{K}$ 07/11/2022

Def. A representation of A in a vector space V :

$$\rho: A \longrightarrow \text{End}(V) \quad \text{homomorphism of algebras.}$$
$$1_A \longmapsto \text{Id}_V.$$

Subrep., quotient rep, direct sum, indec., irreducible
reps are defined in the standard way.

Irrep of A : V s.t. each subrep of V is
either 0 or V .

Morphism $V \xrightarrow{\varphi} W$: $\rho_W(a) \circ \varphi = \varphi \circ \rho_V(a) \quad \forall a \in A.$

Lemma. (Schur). Let V, W be irreps of A .

(i) $\varphi: V \rightarrow W$ is either 0 or an isomorphism;

(ii) $\text{End}(V)$ form a division algebra over K ;

(iii) If K alg. closed, $\text{End}(V) = K$.
(every $\varphi = \lambda \cdot \text{Id}$).

Proof: (i) $\text{Ker } \varphi \subset V$ and $\text{Im } \varphi \subset W$ are subreps;
If $\varphi \neq 0$, then $\text{Ker } \varphi \neq V, \Rightarrow \text{Ker } \varphi = 0$
 $\text{Im } \varphi \neq 0 \Rightarrow \text{Im } \varphi = W$.

So $\varphi: V \xrightarrow{\sim} W$.

(ii). $\text{End } V$ form an algebra over K ;
all nonzero elements of it are isom. $\Rightarrow$
 $\Rightarrow$ invertible.

(iii) $\varphi - \lambda \text{Id}$ has nonzero kernel for some $\lambda \in K$.

so $\varphi - \lambda \text{Id} = 0, \varphi = \lambda \cdot \text{Id}. \quad \square$

Non-example: if $K = \mathbb{R}$, $A = \mathbb{C}$, $V = A = \mathbb{C}$.
 $\text{End}(V) \simeq \mathbb{C}$

Def. A rep. is called semisimple (completely reducible) if it is a direct sum of irreps.

A fin. dim'l algebra A is semisimple if all its f.d. reps. are semisimple.

Example: If $|G| < \infty$, $\mathbb{C}G$ is semisimple (Maschke's thm)
if $Q = (\bullet \rightarrow \bullet)$ then P_Q is not semisimple.

Example. If V is irred, $Y = \text{End}(V)$ is acted on by A .
This rep. is semisimple: $Y \simeq nV$, $n = \dim V$.
Indeed, let $v_1, \dots, v_n$ be a basis in V .

$$\text{End } V \longrightarrow nV$$

$$x \longmapsto (xv_1, xv_2, \dots, xv_n).$$

Let K be alg
closed.

Prop. let V_i , $i=1, \dots, m$, be a set
of pairwise distinct irreps of A ,

$$V = n_1 V_1 \oplus \dots \oplus n_m V_m.$$

W subrep.

Then $W \simeq z_1 V_1 \oplus \dots \oplus z_m V_m$, $z_i \leq n_i$.

$$\varphi: W \longrightarrow V: \quad \varphi = \bigoplus \varphi_i, \quad \varphi_i: z_i V_i \longrightarrow n_i V_i$$

φ_i is given by mult. by an $z_i \times n_i$ matrix with
linearly indep. rows.

$$\varphi_i(v_1, \dots, v_z) = (v_1, \dots, v_z) X_i.$$

$$X_i \in \text{Mat}(z_i \times n_i, K)$$

Proof. Induction in $n = \sum n_i$. Base is obvious.

Ind. step: $W \neq 0$, $P \subset W$ irred. subrep.

[Exercise: show that if V is not necess. f.dim'l, every subrep W contains an irred. subrep P].

Then P is isom. to some V_i .

$$\begin{array}{ccccc} \varphi|_P: P & \xrightarrow{\quad} & V & \xrightarrow{\pi_i} & n_i V_i \\ \downarrow \text{is} & & & & \\ V_i & \xrightarrow{\quad} & & & \end{array} \quad \lambda_i \in K)$$

$$v \mapsto (v \lambda_1, \dots, v \lambda_{n_i}), \quad \text{not all } \lambda_i \text{'s are } 0.$$

$$G_i = GL(n_i, K) \curvearrowright n_i V_i.$$

$$(v_1, \dots, v_{n_i}) \mapsto (v_1, \dots, v_{n_i}) g_i \quad (\text{acts on the remaining comp. by identity}).$$

$$x_i \mapsto x_i g_i, \quad \text{and the remaining } x_j, \quad j \neq i$$

$$\text{Take } g_i: (\lambda_1, \dots, \lambda_{n_i}) g_i = (1, 0, \dots, 0) \quad \text{do not change.}$$

$$P \mapsto P \cdot g_i = (V_i \oplus 0 \oplus \dots \oplus 0) \subset n_i V_i.$$

$$W g_i = V_i \oplus W', \quad W' \subset n_1 V_1 \oplus \dots \oplus (n_i - 1) V_i \oplus \dots \oplus n_m V_m.$$

$$W' = \text{Ker} \left(W g_i \rightarrow (V_i \oplus 0 \oplus \dots \oplus 0) \right)$$

The density theorem.

$$\overline{K} = K.$$

Thm. (Density theorem Lite). V irrep of A ,
 $v_1, \dots, v_n \in V$ lin. indep.

Then for any $w_1, \dots, w_n \in V$ $\exists a \in A$:
 $av_i = w_i$

Proof Assume the converse.

$A \rightarrow nV$ - semisimple rep.

$$a: (v_1, \dots, v_n) \mapsto (av_1, \dots, av_n).$$

If $A \rightarrow uV$ is proper, it corr. to some $z \times n$ matrix, X
 $z < n$.

$a=1$: $\exists u_1, \dots, u_r \in V$:

$$(u_1, \dots, u_r) X = (v_1, \dots, v_n)$$

Since $z < n$, $\exists (\lambda_1, \dots, \lambda_n)$: $X(\lambda_1, \dots, \lambda_n)^T = 0$. $(\lambda_1, \dots, \lambda_n) \neq (0, \dots, 0)$

$$\sum_{i=1}^n \lambda_i v_i = (u_1, \dots, u_r) X (\lambda_1, \dots, \lambda_n)^T = 0. \quad \text{contr. with lin. indep. } \square$$

Thm. (i) Let V be a f.d. irrep of A .

Then $\rho: A \rightarrow \text{End } V$ is surjective.

(ii) $V = V_1 \oplus \dots \oplus V_r$, $V_i \not\cong V_j$.

Then $\rho = \bigoplus \rho_i: A \rightarrow \bigoplus \text{End } V_i$ is surjective.

Proof. (i) $B = \text{Im } \rho \subset \text{End } V$.

$c \in \text{End } V$, $v_1, \dots, v_n$ basis of V

$$w_i = cv_i.$$

$\exists a \in A$: $w_i = av_i$. So $\rho(a) = c$.

So any $c \in \text{End } V$ is in $\text{Im } \rho$ qed.

(ii) Let B_i be the image of A in $\text{End}(V_i)$

$B \quad \text{---} \text{''} \text{---} \quad A \text{ in } \bigoplus \text{End}(V_i)$

As a rep. of A , $\bigoplus \text{End } V_i \simeq \bigoplus d_i V_i$, $d_i = \dim V_i$.

Then $B = \bigoplus B_i$. But due to (i) $\text{End } V_i = B_i$. $\square$

Reps of sums of matrix algebras.

$$A = \bigoplus_{i=1}^r \text{Mat}_{d_i}(\mathbb{K}).$$

Thm. The irreps of A are $V_1 = \mathbb{K}^{d_1}, \dots, V_r = \mathbb{K}^{d_r}$.

Any f.d. rep. of A is a direct sum of V_i 's with certain multiplicities.

Def. V is a rep. of A .

the dual rep. V^* is the rep. of A^{op} , def. by:

$$(f \cdot a)(v) := f(av).$$

[A^{op} : A as a vector space, but $a \cdot^{\text{op}} b = b \cdot a$].

Remark: $\text{Mat}(d, \mathbb{K})^{\text{op}} \simeq \text{Mat}(d, \mathbb{K})$: $A \mapsto A^T, (AB)^T = B^T A^T$.

Proof of Thm. V_i are irreducible.

Let X be an n -dim'l rep of A

X^* ———— of $A^{op} \simeq A$.

$y_1, \dots, y_n \in X^*$ $\varphi: \overbrace{A \oplus \dots \oplus A}^n \rightarrow X^*$

$\varphi(a_1, \dots, a_n) = a_1 y_1 + \dots + a_n y_n$ surjective

$\varphi^*: X \rightarrow A^* \oplus \dots \oplus A^* \simeq A \oplus \dots \oplus A$ is then injective.

$$\left(\bigoplus_i \dim V_i \right)^{\oplus n} = \bigoplus_i \dim V_i.$$

So $X = \bigoplus m_i V_i$, as desired.

Filtrations.

A algebra, V its representation

Def. A filtration of V is a sequence of subsp

$$0 = V_0 \subset V_1 \subset \dots \subset V_n = V.$$

Lemma. Any finite dim'l rep V admits a finite filtration s.t. V_i/V_{i-1} are irred.

Proof By induction on $\dim V$. Base: obvious.

$V_1 \subset V$ irred; consider $U = V/V_1$.

$0 \subset U_0 \subset U_1 \subset U_2 \subset \dots \subset U_{n-1} = U$ exists by the induction assumption

U_i/U_{i-1} are irred

$$\pi: V \rightarrow V/V_1 = U$$

$$\text{define } V_i = \pi^{-1}(U_{i-1}), \quad i \geq 2.$$

Then $V_1 \subset V_2 \subset \dots \subset V_n$ has the desired property. $\square$

NO CLASS this Thursday, 10/11/22!