

Proof of Gabriel's theorem.

Lecture 16

3/11/2022

Thm. A quiver Q is of fin. type $\Leftrightarrow$

$\Leftrightarrow$ its graph is A_n, D_n , or $E_{6,7,8}$. There is a bijection between indec. reps of Q and positive roots of the corresp root system;

$$\underline{V}, \underline{\dim V} = (d_1, \dots, d_n) \iff R^+ \ni \alpha = \sum_{i=1}^n d_i \alpha_i.$$

Last time: Reflection functors

$$\text{Rep } Q \xrightarrow{F_i^+} \text{Rep } \overline{Q}_i, \quad i \text{ is a sink}$$



$$\text{Rep } Q \xrightarrow{F_i^-} \text{Rep } \overline{Q}_i \quad i \text{ is a source.}$$

(1) If V indec., then either $F_i^+ V$ indec.,

or $F_i^+ V = 0, \quad V = S_i$

Same about F_i^- .

(ii) If $\varphi: \bigoplus V_j \rightarrow V_i$ is surjective, (i sink)

$$F_i^- F_i^+ V = V.$$

If $\psi: V_i \rightarrow \bigoplus V_j$ injective, (i source,

$$F_i^+ F_i^- V = V.$$

(iii) If $\varphi: \bigoplus V_j \rightarrow V_i$ is surjective, (i sink)

$$\dim(F_i^+ V) = s_i(\dim V).$$

If $\psi: V_i \rightarrow \bigoplus V_j$ injective, (i source,

$$\dim(F_i^- V) = s_i(\dim V).$$

Coxeter elements. Let Q be a Dynkin quiver

let $\{1, 2, \dots, n\}$ be any labeling of vertices

$c = s_1 \dots s_n$ a Coxeter element.

Lemma. Let $0 \neq \beta = \sum k_i \alpha_i$, $k_i \geq 0$. $\exists N \in \mathbb{N}$:

$c^N \beta$ has at least one negative coef. ($\Rightarrow c^N \beta$ negative)

Proof. $W = \langle s_1, \dots, s_n \rangle \quad |W| < \infty \quad \exists M: C^M = I.$

Claim: As an operator on $\mathbb{R}^n$, $I + C + C^2 + \dots + C^{M-1} = 0.$

$0 = (I + C + \dots + C^{M-1})\beta = \beta + C\beta + \dots + C^{M-1}\beta$, So some of the coef
of some $C^k\beta$ is < 0 .

$$I + C + \dots + C^{M-1} = (I - C^M) \cdot (I - C)^{-1}.$$

Need to show: I is not an eigenvalue for C ,
hence $C - I$ is invertible.

Suppose $Cv = v$.

$$s_1 \dots s_n v = v.$$

$$s_2 \dots s_n v = s_1 v$$

$$s_1 v = v$$

$$s_2 \dots s_n v = v.$$

s_1 changes only the 1st
coord. of $v \Rightarrow$

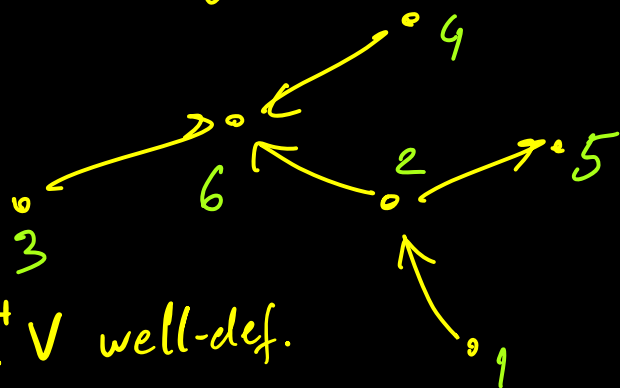
$$B(v, \alpha_i) = 0, \quad v \perp \alpha_i \Rightarrow v = 0.$$

Proof of Gabriel's theorem.

Label the vertices of Q as follows:

$i < j$, if there exists a path $i \rightarrow j$.

(such a labeling exists for every oriented tree):



$F_1^+ \dots F_n^+ V$ well-def.

In particular, n is a sink.

$$F_1^+ \dots F_n^+ = \Phi^+ : \text{Rep } Q \rightarrow \text{Rep } Q$$

Let V be an indep. rep. of Q .

$$V^{(0)} = V, \quad V^{(1)} = F_n^+ V, \quad V^{(2)} = F_{n-1}^+ F_n^+ V, \quad \dots \quad V^{(n+1)} = F_1^+ V^{(n)}$$

Claim: this is a well-defined sequence!

Thm. $\exists m \in \mathbb{N} : \dim V^{(m)} = \alpha_p \quad [V^{(m)} = S_p]$ for some p .

Proof. If $V^{(i)}$ is surjective at k , $k \equiv i \pmod n$, then

$$\dim(V^{(i+1)}) = \dim(F_k^+ V^{(i)}) = s_i(\dim V^{(i)}).$$

$$\dim V^{(i)} = s_i \dots s_n \dim V.$$

$$\Phi^+ = F_1^+ \dots F_n^+.$$

Claim: $(\Phi^+)^N V = 0$ for $N \in \mathbb{N}$.

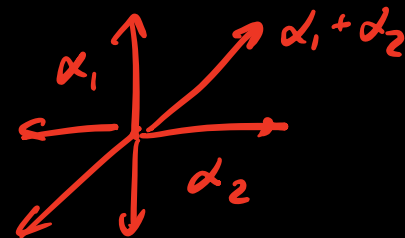
This means that for some i , $V^{(i+1)} = 0$.

$$V^{(i+1)} = F_k^+ V^{(i)} \leftarrow \text{not surjective at } k. \Rightarrow$$

$$\Rightarrow V^{(i)} = S_k.$$

$$\alpha_i \mapsto -\alpha_i$$

$$s_i : \mathbb{R}^+ \setminus \{\alpha_i\} \rightarrow \mathbb{R}^+ \setminus \{\alpha_i\}$$



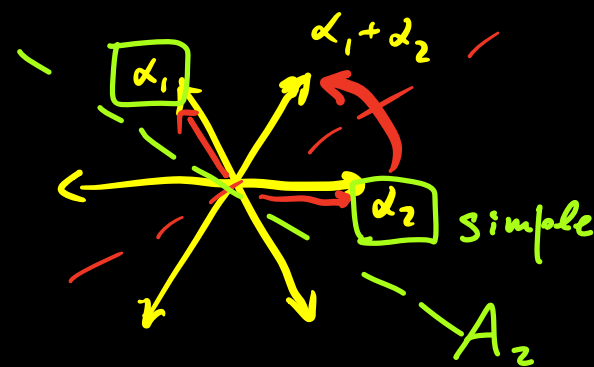
Sidebar: an important lemma about root systems.

$$R \supset R^+ \supset \Delta = \{\alpha_1, \dots, \alpha_n\} \leftarrow \text{basis in } \mathbb{R}^n$$

Prop: $s_i: \alpha_i \mapsto -\alpha_i$

$$R^+ \setminus \{\alpha_i\} \rightarrow R^+ \setminus \{\alpha_i\}$$

Proof: $\beta \in R^+, \beta \neq \alpha_i \Rightarrow \beta = \sum_{j=1}^n c_j \alpha_j$
 $\exists j: c_j > 0.$



$$\beta - B(\alpha_i, \beta)\alpha_i = s_i \beta = \sum_{j \neq i} c_j \alpha_j + \tilde{c}_i \alpha_i -$$

the j -th coord. is positive

$$\Rightarrow s_i \beta \in R^+. \quad \square$$

Thm. (P. Gabriel, 1972). Q Dynkin quiver, V indec. rep. Then $\dim V$ is a positive root.

Proof. $s_{i_1} \dots s_{i_m}(\dim V) = \alpha_p$

$$\Leftrightarrow F_{i_1}^+ \dots F_{i_m}^+ V = S_p$$

$$B(\dim V, \dim V) = B(\alpha_p, \alpha_p) = 2.$$

Corollary Let $V, \tilde{V}$ be indec. rep., $\dim V = \dim \tilde{V}$.
Then $V \simeq \tilde{V}$.

Proof.

Suppose $F_{i_1}^+ \dots F_{i_N}^+ V = S_p$

$$s_{i_1} \dots s_{i_N}(\dim V) = \alpha_p.$$

$$s_{i_1} \dots s_{i_N}(\dim \tilde{V}) = \alpha_p.$$

So $F_{i_1}^+ \dots F_{i_N}^+ \tilde{V}$ is indec with $\dim \alpha_p \Rightarrow S_p$.

$$F_{i_1}^+ \dots F_{i_N}^+ \tilde{V} \cong S_p \simeq F_{i_1}^+ \dots F_{i_N}^+ V,$$

$$F_{i_N}^- \dots F_{i_1}^- (F_{i_1}^+ \dots F_{i_N}^+) \tilde{V} \simeq F_{i_N}^- \dots F_{i_1}^- (F_{i_1}^+ \dots F_{i_N}^+) V$$

$$\stackrel{||}{\simeq} \tilde{V}$$

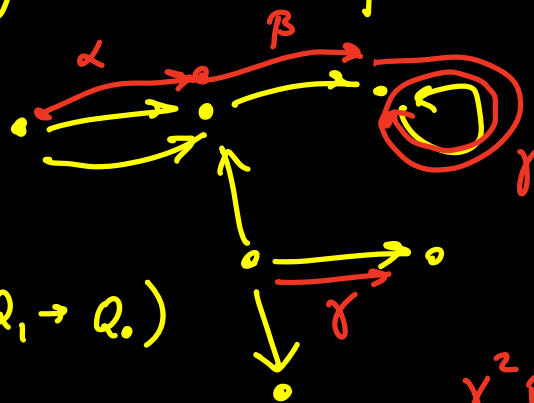
$$\simeq$$

$$\stackrel{||}{\simeq} V$$

q.e.d. $\square$

Path algebra of a quiver.

Q



$$A_Q = \langle e_\alpha, e_i \mid \begin{matrix} \alpha \in Q_1 \\ i \in Q_0 \end{matrix} \rangle_{\mathbb{C}}$$

$$Q = (Q_0, Q_1, s, t: Q_1 \rightarrow Q_0)$$

$\gamma^2 \beta \alpha$ - path.

Multiplication of paths = concatenation:

$$\rho \circ \pi =$$

$$\beta_m \dots \beta_1 \cdot \alpha_n \dots \alpha_1 = \begin{cases} \beta_m \dots \beta_1 \alpha_n \dots \alpha_1, & \text{if } s(\beta_1) = t(\alpha_n) \\ & s(\rho) = t(\pi). \\ 0, & s(\rho) \neq t(\pi). \end{cases}$$

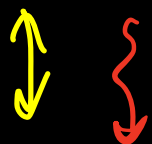
$$e_i^2 = e_i, \quad e_i e_j = 0 \quad i \neq j, \quad \alpha e_i = \begin{cases} \alpha, & \text{if } s(\alpha) = i \\ 0 & \text{otherwise} \end{cases}$$

$$e_j \alpha = \begin{cases} \alpha, & \text{if } t(\alpha) = j \\ 0 & \text{otherwise.} \end{cases}$$

$$1 = e_1 + \dots + e_n$$

A_Q associative $\mathbb{C}$ -algebra with 1.

Rep. of A_Q : $\rho: A_Q \rightarrow \text{End}(V)$.



Same as a repr. of Q .

$\pi_i: V \rightarrow V_i$ - projectors onto V_i 's

$$v \mapsto e_i v$$

$$V = V_1 \oplus \dots \oplus V_n.$$

$$i \xrightarrow{\alpha} j \quad \longrightarrow \quad \varphi_\alpha(v_i) = e_\alpha \cdot v_i, \quad v_i \in V_i,$$

$\Downarrow$
 V_j

$$e_j e_\alpha v_i = e_{j\alpha} v_i = e_\alpha v_i.$$

Opposite direction: $V = \bigoplus V_i, \quad e_i(v) = \pi_i v,$

$$e_\alpha v = \varphi_\alpha \pi_i(v)$$

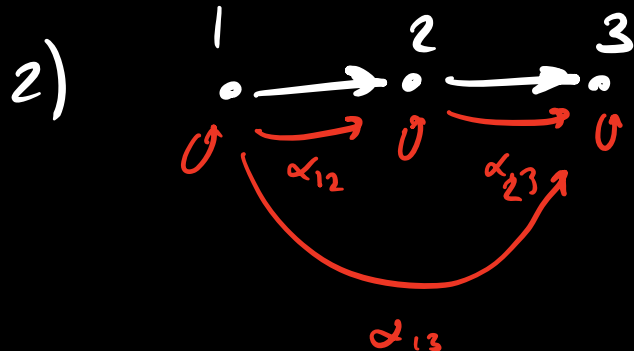
Examples: 1) $\overset{1}{\circ} \longrightarrow \overset{2}{\circ}$

$$\dim_{\mathbb{C}} A_Q = 3$$

$$A_Q = \mathbb{C}\langle e_1, e_2, e_{12} \rangle$$

$$\begin{array}{l} / e_1^2 = e_1 \\ e_2^2 = e_2 \end{array}$$

$$e_2 e_{12} = e_{12} e_1 = e_{12}$$



$$\dim A_Q = 6$$

3) $\circ \longrightarrow \circ \longleftarrow \circ \quad \dim A_Q = 5$

4) $\underset{1}{\circ} \circlearrowright \alpha$

$$A_Q = \langle 1, \alpha, \alpha^2, \dots \rangle_{\mathbb{C}} \simeq \mathbb{C}[x]$$

5)  $A_Q = \mathbb{C}\langle x, y \rangle$.

Prop. $|\dim A_Q| < \infty \Leftrightarrow Q$ has no oriented cycles (incl. loops).

Next time: repr. of finite-dim'l algebras.

No class on Thu, Nov. 10!

Last lecture: Dec. 5, Mon.