

Γ Dynkin graph $\rightarrow q_r = 2 \left(\sum_{i \in \Gamma} x_i^2 - \sum_{i \sim j} x_i x_j \right)$ Lecture 15
 $(A_n, D_n, E_{6,7,8})$ 31/10/2022

This is a positive definite form. $q_r(v) \in 2\mathbb{Z}$.

B_r sym. bilinear form assoc. to q_r .

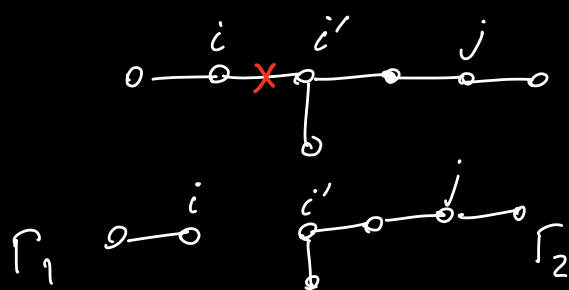
Def. A root $\alpha \in \mathbb{Z}^n$ is a vector s.t. $q_r(\alpha) = 2$.

simple roots: $\alpha_i = (0, \dots, 0, 1, 0, \dots, 0)$

$R \subset \mathbb{Z}^n$ the root system, $\Delta = \{\alpha_1, \dots, \alpha_n\} \subset R$ simple roots

Lemma. Let $\alpha \in R$, $\alpha = \sum k_i \alpha_i$, all $k_i \geq 0$ or all $k_i \leq 0$.

Proof: Assume $\alpha = \sum k_i \alpha_i$, $k_i > 0$, $k_j < 0$. Suppose $k_\ell = 0$



Remove an edge adj to i , $i < \ell < j$.
 s.t. i, j are in different components.

$i \text{ --- } i'$

$$\beta = \sum_{m \in I_1} k_m \alpha_m$$

$$\gamma = \sum_{m \in I_2} k_m \alpha_m$$

$$\alpha = \beta + \gamma. \quad \beta \neq 0, \quad \gamma \neq 0.$$

$$q_r(\alpha) = q_r(\beta) + q_r(\gamma) + 2B(\beta, \gamma).$$

$$k_i > 0$$

$$k_{i'} \leq 0$$

$$B(\beta, \gamma) = -2k_i k_{i'}$$

$$q_r(\alpha) \geq 4 \quad \text{But } \alpha \text{ is a root.}$$

Contradiction.

□

$$R = R^+ \sqcup R^- \quad \text{positive and negative roots.}$$

Example. $A_{n-1} \subset L = \{(x_1, \dots, x_n) \mid \sum x_i = 0\} \subset \mathbb{Z}^n$

$$\alpha_1 = (1, -1, 0, \dots, 0)$$

$$\alpha_2 = (0, 1, -1, 0, \dots, 0)$$

$$\alpha_{n-1} = (0, \dots, 0, 1, -1).$$

$$(\alpha_i, \alpha_j) = \begin{cases} 2, & i=j \\ -1, & |i-j|=1 \\ 0, & \text{otherwise.} \end{cases} \quad \begin{pmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{pmatrix}$$

$$\alpha_1 \quad \alpha_2 \quad \dots \quad \alpha_{n-1}$$

$$R^+ = \{ (0, \dots, 0, 1_i, 0, \dots, 0, -1_j, \dots, 0) \} = \alpha_{ij} = \alpha_i + \alpha_{i+1} + \dots + \alpha_{j-1} \quad i < j$$

$$|R^+| = \frac{n(n-1)}{2} = \binom{n}{2}.$$

$$\begin{array}{ccccccccccc} 0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 \\ 0 & \rightarrow & \mathbb{C} & \rightarrow & \mathbb{C} & \rightarrow & \mathbb{C} & \rightarrow & 0 & \rightarrow & 0 \end{array} \quad V_{ij}$$

$$(0, \dots, 1, \dots, 1, 0, \dots, 0)$$

Def. Let $\alpha \in R^+$ be a positive root. Then define s_α

$$v \in \mathbb{Z}^n \quad s_\alpha(v) = v - B(v, \alpha)\alpha. \quad \text{reflections}$$

$$s_\alpha: \alpha \mapsto -\alpha, \quad s_\alpha v = v \text{ for } B(v, \alpha) = 0.$$

Simple reflections:

$$s_i = s_{\alpha_i}$$

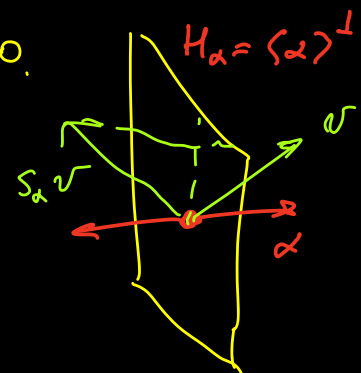
v and $s_i v$ differ only in the i th coordinate.

won't prove this now...

$$\text{Hermann} \quad W = \langle s_i \mid i=1, \dots, n \rangle = \langle s_\alpha \mid \alpha \in R^+ \rangle$$

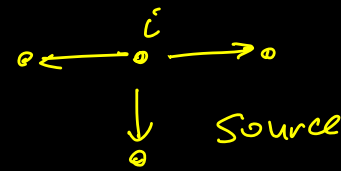
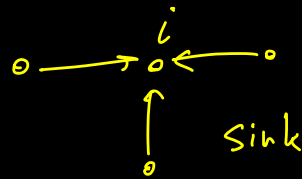
Weyl

$O(n)$ $W \curvearrowright R$ faithfully. W is finite.



Reflection functors

Def. Q quiver
 i sink or source



$Q \xrightarrow{\text{reflection}} \overline{Q}_i$ the arrows adjacent to i reversed.

Exercise. Let Q be an acyclic quiver.

Then Q can be brought by reflections to $\widehat{Q}$ with any orientation of arrows.

Let $Q \ni i$ be a sink, let $V \in \text{Rep } Q$.

$$F_i^+: \text{Rep } Q \longrightarrow \text{Rep } \overline{Q}_i \quad \longrightarrow V_i \xleftarrow{\varphi_{ji}} V_j$$

$$F_i^+(V)_k = V_k \quad k \neq i$$

$$F_i^+(V)_i = \text{Ker} \left(\varphi: \bigoplus_{j \rightarrow i} V_j \longrightarrow V_i \right)$$

$$\varphi: \bigoplus V_j \longrightarrow V_i$$

$$\varphi(\sum v_j) = \sum \varphi_{ji}(v_j)$$

φ_{ji} are being replaced by

$$\text{Ker}(\varphi) \hookrightarrow \bigoplus V_j \xrightarrow{\pi_j} V_j$$

$$\begin{array}{ccc} \leftarrow \text{Ker } \varphi & \longrightarrow & V_j \\ \downarrow & & \end{array}$$

$$\begin{array}{ccc} V_k & \longleftarrow & V_i \longrightarrow V_j \\ \downarrow & & \downarrow \\ & U_k & \end{array}$$

$$\psi: V_i \longrightarrow \bigoplus_{i \rightarrow j} V_j$$

$Q \ni i$ source
 $F_i^-: \text{Rep } Q \longrightarrow \text{Rep } \overline{Q}_i$

$$F_i^-(V)_k = V_k$$

$$F_i^-(V)_i = \text{Coker } \psi = \left(\bigoplus_{i \rightarrow j} V_j \right) / \text{Im } \psi$$

Maps: $V_j \xleftrightarrow{\ell_j} \bigoplus_{i \rightarrow j} V_j \longrightarrow \bigoplus V_j / \text{Im } \psi$. $V_k \rightarrow \text{Coker } \psi \leftarrow V_j$
 $\uparrow$
 V_k

Example: i sink, S_i "simple rep": $(S_i)_j = 0$

$$\begin{array}{ccc} 0 \longrightarrow \mathbb{C} \longleftarrow 0 & \xrightarrow{F_i^+} & 0 \longleftarrow 0 \longrightarrow 0 \\ \uparrow & \searrow & \downarrow \\ 0 & & 0 \end{array}$$

$$(S_i)_i = \mathbb{C}$$

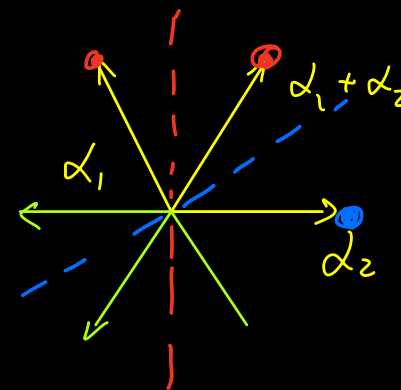
$$F_i^+ S_i = 0.$$

Example: $\overset{1}{0} \longrightarrow \overset{2}{0} \quad A_2 \quad S_1 = (\mathbb{C} \rightarrow 0) \quad \alpha_1$

$S_2 = (0 \rightarrow \mathbb{C}).$

$(\mathbb{C} \rightarrow 0) \xrightarrow{F_2^+} \underbrace{\mathbb{C} \xleftarrow{\text{Id}} \mathbb{C}}$

$F_1^+(\mathbb{C} \xleftarrow{\text{Id}} \mathbb{C}) = (0 \rightarrow \mathbb{C}) = S_2$



Prop. (i) Let $Q \ni i$ be a sink, and let V index.

Then either $\varphi: \bigoplus V_j \rightarrow V_i$ is surjective

or $V \simeq S_i \quad (V_i = \mathbb{C}, V_k = 0 \quad k \neq i).$

(ii) Let $Q \ni i$ be a source, V index.

Then either $\psi: V_i \rightarrow \bigoplus V_j$ injective or

$V \simeq S_i.$

Proof: (i) $\varphi: \bigoplus V_j \rightarrow V_i$ $V_\ell \rightarrow V_i \leftarrow V_j$
 W complement to $\text{Im } \varphi$. $\uparrow$
 V_k

$$V = V_\ell \rightarrow \text{Im } \varphi \leftarrow V_j \quad \bigoplus \quad 0 \rightarrow W \leftarrow 0$$

$\uparrow$ $\uparrow$
 V_i 0

So either $W = 0$, or $\uparrow = 0$, and $W = \mathbb{C}$. ($\Rightarrow V = S_i$).

(ii) similarly. $\square$

Prop. (i) If $\varphi: \bigoplus_{j \rightarrow i} V_j \rightarrow V_i$ is surjective, (i sink)
then $F_i^- F_i^+ V = V$.

(ii) If $\psi: V_i \rightarrow \bigoplus_{i \rightarrow j} V_j$ is injective (i source),
then $F_i^+ F_i^- V = V$.

Proof. Need to show: $F_i^- F_i^+ V = V$

$$\varphi: \bigoplus_{j \rightarrow i} V_j \rightarrow V_i \quad \text{surjective}$$

$$\text{Ker } \varphi = K.$$

$$(F_i^+ V): \quad V_i \longrightarrow K$$

$$\psi: K \longrightarrow \bigoplus V_j \quad \text{injective}$$

F_i^- : replaces K by $K' = (\bigoplus V_j) / \text{Im } \psi$. the isom. theorem

$$\text{Im } \psi \simeq K$$

$$K' = (\bigoplus V_j) / \text{Ker}(\varphi: \bigoplus V_j \rightarrow V_i) \stackrel{\text{red arrow}}{=} \text{Im}(\varphi: \bigoplus V_j \rightarrow V_i)$$

$\downarrow$
 V_i

$K' = V_i$ because φ is surjective
the maps are also the same.

(ii) Similarly. $\square$

Prop. Let V be indecomposable. Then $F_i^+ V$, $F_i^- V$ are either 0 or indec. (whenever exist).

Proof For F_i^+ : $\varphi: \bigoplus_{j \rightarrow i} V_j \rightarrow V_i$ is surj or $V = S_i$.

Assume $F_i^+ V = X \oplus Y$. But $F_i^+ V$ is inj in i :

$$F_i^- X \neq 0 \quad F_i^- Y \neq 0, \text{ because.}$$

$$F_i^+ F_i^- X = X, \quad F_i^+ F_i^- Y = Y$$

$$V = F_i^- F_i^+ V = F_i^- X \oplus F_i^- Y. \quad \text{decomposable.} \quad \square$$

Prop. Let Q be a quiver, V indec. Q -rep.

(i) if $i \in Q$ is a sink, and V is surjective at $i \Rightarrow$

$$\Rightarrow \dim(F_i^+ V) = s_i(\dim V)$$

(ii) $i \in Q$ source, V inj. at i $\dim(F_i^- V) = s_i(\dim V)$.

Proof: (i) $S_i v = v - B(v, \alpha) \alpha$.

$\varphi: \bigoplus_{j \rightarrow i} V_j \rightarrow V_i$ surjective.

$$\dim(F_i^+ V)_i = \dim \text{Ker } \varphi = \sum_{j \rightarrow i} \dim V_j - \dim V_i$$

$$\dim(F_i^+ V)_i - \dim V_i = \sum_{j \rightarrow i} \dim V_j - 2 \dim V_i = -B(\dim V, \alpha_i)$$

$$\dim F_i^+ V = \dim V - B(\dim V, \alpha_i) = S_i(\dim V). \quad \square$$