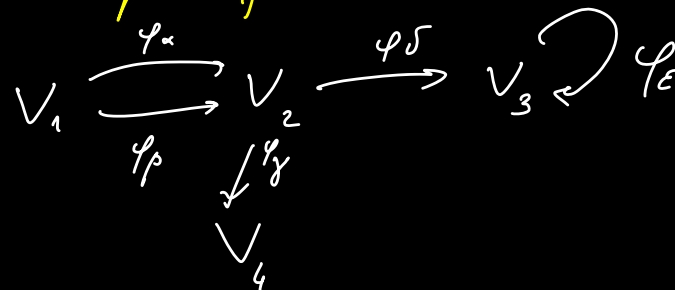


Gabriel's theorem

Lecture 14

27/10/22

Q quiver $\leadsto V$ rep. of Q



Indecomposable $\neq$ irreducible!

Def. Q is of finite type if it has fin. many indec. reps.

Thm. (Gabriel '72) $\checkmark$ A connected quiver Q is of finite type iff its graph is one of the following:

$\circ - \circ - \dots - \circ \quad A_n \quad n \geq 1$

$\begin{array}{c} \circ \\ \diagup \end{array} \circ - \dots - \circ \quad D_n \quad n \geq 4$

$\circ - \circ - \circ \begin{array}{c} \circ \\ | \end{array} \circ - \circ \quad E_6$

$\circ - \circ - \circ \begin{array}{c} \circ \\ | \end{array} \circ - \circ - \circ \quad E_7$

$\circ - \circ - \circ \begin{array}{c} \circ \\ | \end{array} \circ - \circ - \circ - \circ \quad E_8$

First goal: prove the "only if" part.

Some alg. geometry.

Let G be an alg. group acting on an irreducible
(say, $GL(n, \mathbb{C})$) variety X .

Claim. The number of G -orbits on X is finite (say, $\mathbb{A}^N$)
only if there exists an open G -orbit on X .

This can only happen if $\dim G \geq \dim X$.

Let Q be a quiver, $(V_i, \varphi_\alpha) = \underline{V}$ its representation.

$$|Q_0| = n$$

$$\underline{d} = (d_1, \dots, d_n) = \underline{\dim} \underline{V}, \quad d_i = \dim V_i.$$

$$\text{Rep}(\underline{d}) = \bigoplus_{\alpha \in Q_1} \text{Hom}(V_{s(\alpha)}, V_{t(\alpha)}) \cong \bigoplus_{\alpha \in Q_1} \text{Mat}(d_{s(\alpha)} \times d_{t(\alpha)}).$$

$$\begin{array}{ccc} V_i & \xrightarrow{\varphi_\alpha} & V_j \\ g_i \downarrow & & \downarrow g_j \\ V_i & \xrightarrow{\psi_\alpha} & V_j \end{array} \quad \psi_\alpha g_i = g_j \varphi_\alpha$$

$$GL(\underline{d}) = \prod_{i \in Q_0} GL(V_i) \cong \prod_{i \in Q_0} GL(d_i)$$

$$GL(\underline{d}) \curvearrowright \text{Rep}(\underline{d}) \quad g \circ \varphi_\alpha = g_j \varphi_\alpha g_i^{-1}.$$

The action of $Z = Z(GL(\underline{d})) = \{(\lambda E_i : V_i \rightarrow V_i)\}$ acts trivially

$$PGL(\underline{d}) = GL(\underline{d})/Z \text{ acts on } Rep(\underline{d}).$$

Reps are isomorphic $\Leftrightarrow$ they are in the same orbit.

If Q is of finite type, #isoclasses of reps with $\dim \underline{d}$ is finite $\Rightarrow$

$\Rightarrow PGL(\underline{d}) \curvearrowright Rep(\underline{d})$ has fin. many orbits $\Rightarrow$

$$\Rightarrow \dim PGL(\underline{d}) \geq \dim Rep(\underline{d}).$$

$$\sum_{i=1}^n d_i^2 - 1 \geq \sum_{\alpha \in Q_1} d_{s(\alpha)} d_{t(\alpha)}.$$

holds for each $\underline{d}$.

$$(\star) \quad \sum_{i \in Q_0} d_i^2 - \sum_{\alpha \in Q_1} d_{s(\alpha)} d_{t(\alpha)} > 0.$$

Let Γ be the
underlying graph of Q
 Γ_0 vertices
 Γ_1 edges

$$q_\Gamma(d_1, \dots, d_n) = \sum_{i \in \Gamma_0} d_i^2 - \sum_{i \xrightarrow{\alpha} j} d_i d_j$$

(*) equivalent to $q_\Gamma > 0$ positive definite.

Matrix of q_Γ :
$$\begin{pmatrix} 1 & & -\frac{1}{2} \\ & \ddots & \\ -\frac{1}{2} & & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 & & -1 \\ & \ddots & \\ -1 & & 2 \end{pmatrix} =$$

$$= \frac{1}{2} (2\text{Id} - A_\Gamma),$$
 the Cartan matrix A_Γ adjacency matrix of Γ .

If $q_\Gamma > 0$, Γ is admissible.

Classify all admissible graphs!

Subgraph of Γ : subset of vertices + all the edges adjacent to them:



not a subgraph!

1) A subgraph of an admissible graph is admissible.

2) An admissible graph has no mult. edges.

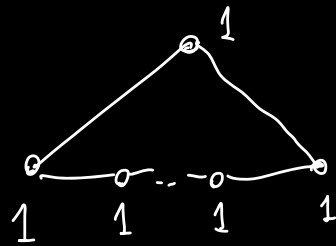


$$\begin{bmatrix} 2 & -a_{ij} \\ -a_{ij} & 2 \end{bmatrix} \neq 0 \text{ if } a_{ij} \geq 2.$$

$$\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$$

3) An admissible graph contains no loops  $[0] \neq 0$.

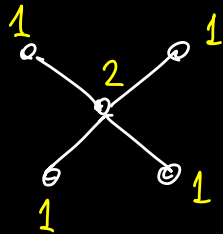
4) An admissible graph has no cycles.



$$q_r(1, \dots, 1) = n - n = 0.$$

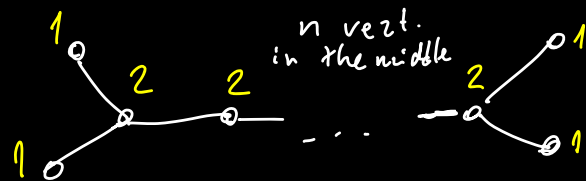
$$\begin{bmatrix} 2 & -1 & & -1 \\ -1 & & & \\ & & & \\ -1 & & & 2 \end{bmatrix}$$

5) An admissible graph has no vertices of degree > 3 .

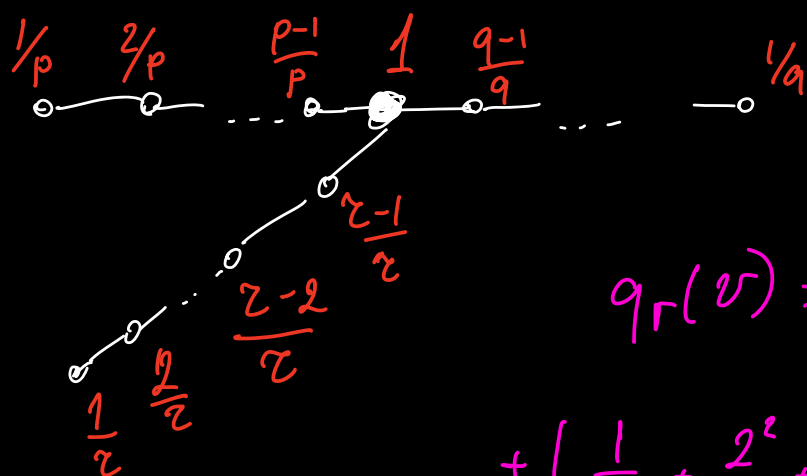


$$q_r = 1^2 + 1^2 + 1^2 + 1^2 + 2^2 - 4 \cdot 2 = 8 - 8 = 0.$$

6) An admissible graph has at most 1 vertex of degree 3.



$$\begin{aligned} q_r(v) &= n \cdot 2^2 + 1 + 1 + 1 + 1 - 4 \cdot 2 - 4 \cdot (n-1) = \\ &= 4(n+1 - 2 - (n-1)) = 0. \end{aligned}$$



Branches of length p, q, z
 $p+q+z-2$ vertices

$$\begin{aligned}
 q_r(v) &= \left(\frac{1}{p^2} + \frac{2^2}{p^2} + \dots + \frac{p^2}{p^2} \right) + \left(\frac{1}{q^2} + \frac{2^2}{q^2} + \dots + \frac{q^2}{q^2} \right) + \\
 &+ \left(\frac{1}{z^2} + \frac{2^2}{z^2} + \dots + \frac{z^2}{z^2} \right) - 2 - \left(\frac{1 \cdot 2}{p^2} + \frac{2 \cdot 3}{p^2} + \dots + \frac{(p-1)p}{p^2} \right) + \\
 &- \left(\frac{1 \cdot 2}{q^2} + \frac{2 \cdot 3}{q^2} + \dots + \frac{(q-1)q}{q^2} \right) - \left(\frac{1 \cdot 2}{z^2} + \dots + \frac{(z-1)z}{z^2} \right) \quad \text{①}
 \end{aligned}$$

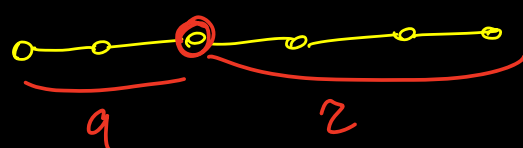
$$\begin{aligned}
 \text{①} - \text{②} &= \frac{1}{p^2} \left(1^2 + \dots + p^2 - 1 \cdot 2 - 2 \cdot 3 - \dots - (p-1) \cdot p \right) = \frac{1}{p^2} (1 + \dots + p) = \\
 &= \frac{1}{p^2} \cdot \frac{p(p+1)}{2} = \frac{p+1}{2p} = \frac{1}{2} + \frac{1}{2p}
 \end{aligned}$$

$$\text{②} = \frac{1}{2} + \frac{1}{2p} + \frac{1}{2} + \frac{1}{2q} + \frac{1}{2} + \frac{1}{2z} - 2 = \frac{1}{2} \left(3 + \frac{1}{p} + \frac{1}{q} + \frac{1}{z} - 4 \right) = \frac{1}{2} \left(\frac{1}{p} + \frac{1}{q} + \frac{1}{z} - 1 \right) > 0$$

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$$

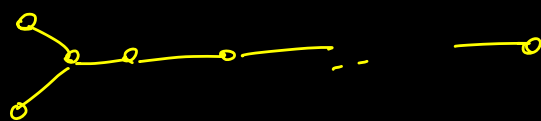
$$p \leq q \leq r$$

Solutions: $(1, q, r)$



$$A_{q+r-1} \sim \mathbb{Z}/h\mathbb{Z}$$

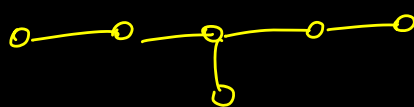
$(2, 2, r)$



$$D_{r+1} \sim \mathbb{D}_n$$

dihedral group

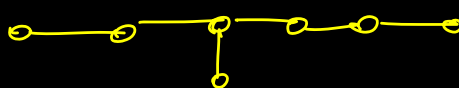
$(2, 3, 3)$



$$E_6$$

symm. of tetrahedron

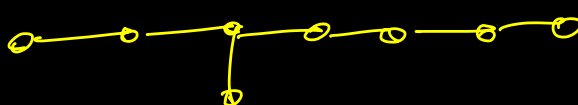
$(2, 3, 4)$



$$E_7$$

cube = octahedron

$(2, 3, 5)$



$$E_8$$

icosahedron = dodecahedron

simply-laced Dynkin diagrams ($\approx$ Coxeter graphs)

Lemma. If $x \in \mathbb{Z}^n$, $2q_{\Gamma}(x)$ is even

$$x^T (2\text{Id} - A_{\Gamma}) x$$

$\mathbb{Z}^n \subset \mathbb{R}^n$ $2q_r$ pos. def. quadratic form.

Def. If $x \in \mathbb{Z}^n$ satisfies $2q_r(x) = 2$, then x is said to be a root.
(the shortest vectors in $\mathbb{Z}^n$).

Example: $x = (0, \dots, 0, 1, 0, \dots, 0) = \alpha_i$ satisfy $2q_r(x) = 2$.
They are called simple roots.

Lemma If $\alpha = \sum k_i \alpha_i$ is a root, then
either $k_i \geq 0$ for each i positive roots $\supset$ simple roots
or $k_i \leq 0$ for each i . negative roots

Thm. (Gabriel). Q is of finite type $\Leftrightarrow q_r > 0$.

In this case, indec. reps $\longleftrightarrow$ positive roots of Γ .