

Representations of quivers

Lecture 13

24.10.22

Quiver = oriented graph

Rep. of quivers = uniform way of treating problems of linear algebras

Q_0 vertices

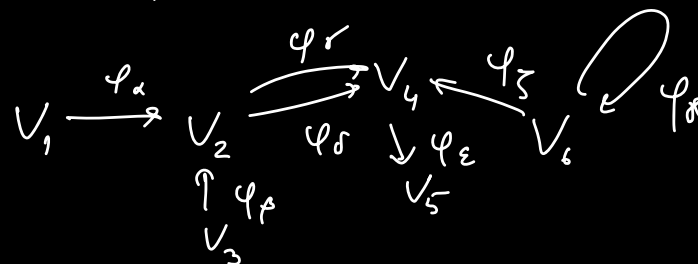
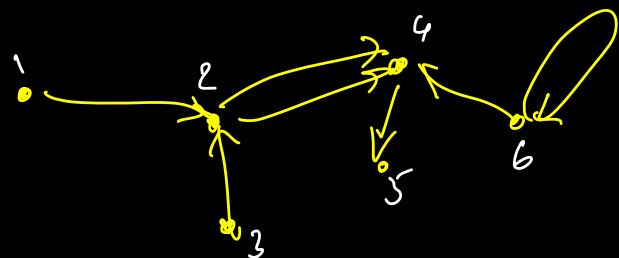
$|Q_0|, |Q_1| < \infty$

Q_1 arrows

Loops and mult arrows are allowed.

$s, t: Q_1 \rightarrow Q_0$

starting and terminal points of an arrow



Def. A representation of a quiver is the following data:

$i \in Q_0 \rightsquigarrow V_i$, $\alpha \in Q_1 \rightsquigarrow \varphi_\alpha: V_{s(\alpha)} \longrightarrow V_{t(\alpha)}$ linear map.

Morphism of reps:
 linear maps
 $F_i: V_i \rightarrow W_i$, s.t.

$$\begin{array}{ccc} V_i & \xrightarrow{\varphi_\alpha} & V_j \\ F_i \downarrow & & \downarrow F_j \\ W_i & \xrightarrow{\psi_\alpha} & W_j \end{array}$$

$$\underline{V} = (V_i, \varphi_\alpha)$$

$$\underline{W} = (W_i, \psi_\alpha)$$

$$F_j \circ \varphi_\alpha = \psi_\alpha \circ F_i.$$

Def. $\underline{V} = (V_i, \varphi_\alpha)$ $\underline{\dim V} = (\dim V_i)_{i \in Q}$ dimension vector.

Isomorphism: F_i must be isomorphisms, and commute with the arrows.

$\text{Ker}, \text{Im}, \dots$ - defined as usual.

Problem: classify reps of Q up to isomorphism.

Some examples:

1)

•

$$V \quad F: V \xrightarrow{\sim} W$$

isoclasses = dim. of V .

$$\begin{array}{ccc} V_1 & \xrightarrow{\varphi} & V_2 \\ F_1 \downarrow & & \downarrow F_2 \\ W_1 & \xrightarrow{\psi} & W_2 \end{array}$$

$\bullet \rightarrow \bullet$

$$\underline{V} \simeq \underline{W}, \text{ if}$$

$$\dim V_1 = \dim W_1,$$

$$\dim V_2 = \dim W_2,$$

$$\varphi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{canonical form}$$

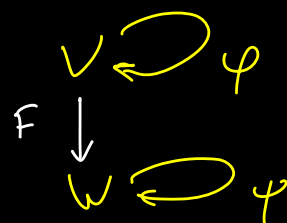
$$\exists F_1 \in GL(V_1), F_2 \in GL(V_2), \quad \varphi = F_2 \circ F_1^{-1}$$

$$\varphi \sim \psi \quad \text{iff} \quad \exists k \quad \varphi = k \psi.$$

Given $\underline{d} = (d_1, d_2)$, the isoclasses of rep. with $\dim \underline{d}$ are given by $\text{rk } \varphi = r$, $0 \leq r \leq \min(d_1, d_2)$.

Gaussian elimination! For each $\underline{d}$, fin. many isoclasses.

2) $\mathcal{O}(V, \varphi)$, $\varphi: V \rightarrow V$



$\varphi \simeq \psi$, if $F \circ \varphi = \psi \circ F$,

F isom. of vector spaces: $\psi = F \varphi F^{-1}$. φ, ψ similar

(define the same linear operator in different coordinates)

Canonical repr. in each class: Jordan normal form.

$$\left[\begin{array}{c|c|c} \lambda_1 & 1 & 0 \\ & \ddots & \\ 0 & & \lambda_1 \\ \hline & & \lambda_2 & 1 & 0 \\ & & & \ddots & \\ 0 & & & & \lambda_2 \\ \hline & & & & & \ddots & \\ & & & & & & \lambda_k & 1 & 0 \\ & & & & & & & \ddots & \\ 0 & & & & & & & & \lambda_k \end{array} \right]$$

Infinitely many isoclasses of each dim!

3)  $(V, \varphi_1, \varphi_2) = \underline{V} \sim \underline{W} = (W, \psi_1, \psi_2)$
if $\exists F: V \xrightarrow{\sim} W: \varphi_1 = F \varphi_1 F^{-1},$

Wild problem. (Too many isoclasses!...) $\psi_2 = F \varphi_2 F^{-1}.$

Irreducible and indec. representations.

Def: $\underline{V}, \underline{W}$ reps of Q .

$$\underline{V} \oplus \underline{W} = \left(V_i \oplus W_i, \quad \underset{i}{\overset{\cdot}{\cdot}} \xrightarrow[\alpha]{\begin{bmatrix} \varphi_\alpha & 0 \\ 0 & \psi_\alpha \end{bmatrix}} \underset{j}{\cdot} \right).$$

$\underline{V}$ is decomposable, if $\underline{V} = \underline{W}_1 \oplus \underline{W}_2$, $W_i \neq 0$,
and indecomposable otherwise.

Def: $\underline{U} \subset \underline{V}$ is a subrep, if $(U_i, \varphi_i) = \underline{U}$,
 $U_i \subset V_i, \quad \varphi_\alpha|_{U_i} = \psi_\alpha: U_i \rightarrow U_j.$

Example: 1) $V_1 \xrightarrow{\varphi} V_2$,

$$\dim V_i = d_i, \quad i=1,2, \\ \operatorname{rk} \varphi = r.$$

$$\varphi \sim \begin{bmatrix} \boxed{1 \dots 1} & 0 \\ 0 & 0 \end{bmatrix}$$

Claim: φ is indecomposable iff

$$d_1, d_2 \leq 1.$$

$$(\mathbb{C} \rightarrow 0)^{\oplus d_1 - r}$$

$$(\operatorname{Ker} \varphi \rightarrow 0) \text{ subrep.} \\ \oplus$$

$$(0 \rightarrow \operatorname{Im} \varphi) \text{ subrep}$$

$$\oplus (0 \rightarrow \mathbb{C})^{\oplus d_2 - r}$$

$$\begin{array}{ccc} \bullet & \xrightarrow{\varphi} & \bullet \\ V_1 & & V_2 \end{array}$$

Indec. reps of $\bullet \rightarrow \bullet : A_2$

$$\mathbb{C} \rightarrow 0$$

irreducible

$$0 \rightarrow \mathbb{C}$$

irreducible

$$\mathbb{C} \xrightarrow{Id} \mathbb{C}$$

indec., but not irreducible!

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} : \operatorname{Coker} \varphi \rightarrow \operatorname{Im} \varphi.$$

$$(\mathbb{C} \rightarrow \mathbb{C})^{\oplus 2}$$

$$0 \rightarrow \begin{bmatrix} 0 \rightarrow \mathbb{C} \\ \text{subrep.} \end{bmatrix} \rightarrow \begin{bmatrix} \mathbb{C} \xrightarrow{1} \mathbb{C} \end{bmatrix} \rightarrow \begin{bmatrix} \mathbb{C} \rightarrow 0 \\ \text{quotient rep.} \end{bmatrix} \rightarrow 0$$

$$[\mathbb{C} \xrightarrow{I_6} \mathbb{C}] \not\sim [\mathbb{C} \xrightarrow{0} \mathbb{C}] \oplus [\mathbb{C} \xrightarrow{0} \mathbb{C}]$$

$$\mathbb{C} \xrightarrow{0} \mathbb{C}$$

Irreducible rep. of (acyclic) quiver: (Exercise). w/o oriented cycles

2) (V, φ) is indecomposable if φ cannot be written as a block matrix.

$\varphi \sim \begin{bmatrix} \lambda & 1 & 0 \\ & \lambda & \vdots \\ 0 & & \ddots & 1 \\ & & & \lambda \end{bmatrix}$ in this basis.

$V = (v_1, v_2, \dots, v_n)$

Subreps = φ -invariant subspaces = $\langle v_1, \dots, v_k \rangle, 0 \leq k \leq n$

Simple: $\langle v_i \rangle, \varphi: v_i \rightarrow \lambda v_i$

Inf. many indecomposables.

Def. Q is of finite type, if it has fin. many indecomposables.

Problem. Classify all finite-type quivers.

3)
$$\begin{array}{c} V_1 \xrightarrow{\varphi_1} V_2 \xrightarrow{\varphi_2} V_3 \\ \circ \longrightarrow \circ \longrightarrow \circ \end{array} \quad A_3 \text{ equioriented}$$

Exercise. Show that this is a complete list of indecomposable reps.

$$\begin{array}{c} \mathbb{C} \longrightarrow 0 \longrightarrow 0 \quad 0 \longrightarrow \mathbb{C} \longrightarrow 0 \quad 0 \longrightarrow 0 \longrightarrow \mathbb{C} \\ \mathbb{C} \xrightarrow{1} \mathbb{C} \longrightarrow 0 \quad 0 \longrightarrow \mathbb{C} \xrightarrow{1} \mathbb{C} \\ \mathbb{C} \xrightarrow{1} \mathbb{C} \xrightarrow{1} \mathbb{C} \end{array}$$

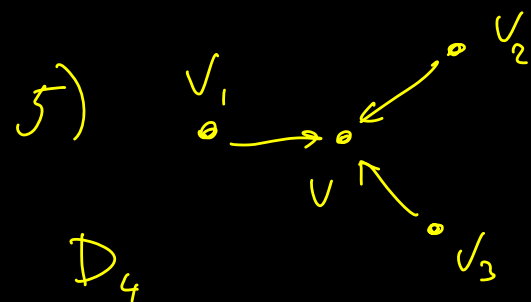
6 indec. reps

4)
$$\begin{array}{c} V_1 \longrightarrow V_2 \longrightarrow \dots \longrightarrow V_n \\ \circ \longrightarrow \circ \longrightarrow \dots \longrightarrow \circ \end{array} \quad A_n \text{ equioriented.}$$

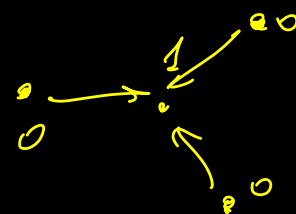
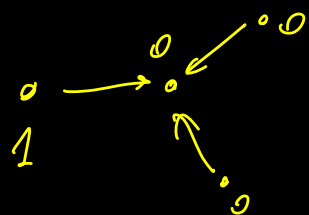
$$0 \longrightarrow \dots \longrightarrow \mathbb{C} \xrightarrow{1} \mathbb{C} \xrightarrow{1} \dots \xrightarrow{1} \mathbb{C} \longrightarrow 0, \dots \longrightarrow 0, \iff (1 \leq i \leq j \leq n).$$

$$1 + 2 + \dots + n = \frac{n(n+1)}{2}.$$

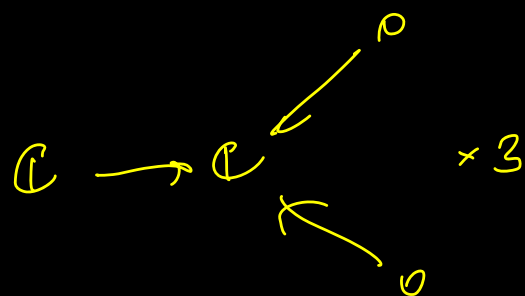
4') Do the same for A_n with any orientation.



$$\varphi_i: V_i \rightarrow V_4, \quad i=1,2,3.$$

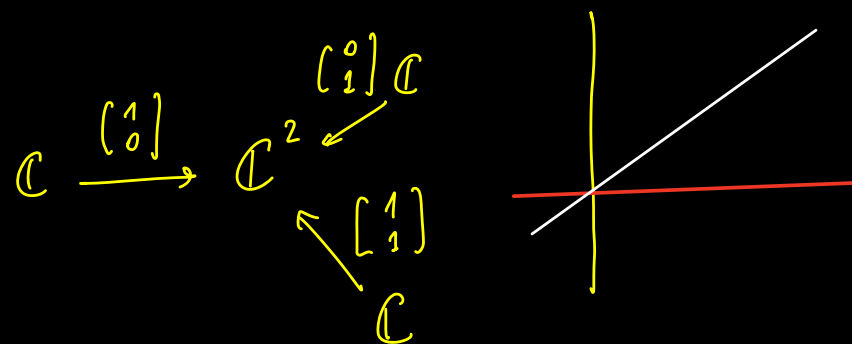
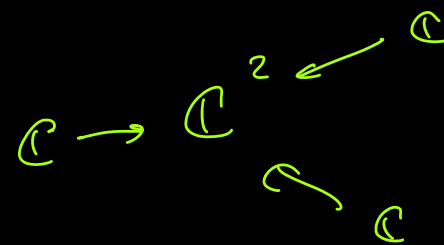
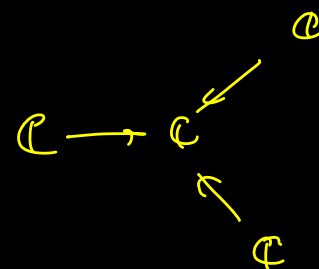
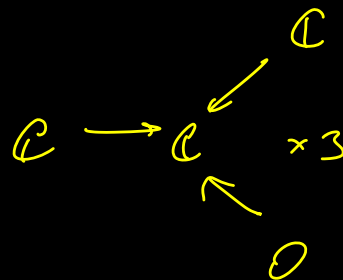


4 irred
 $\Downarrow$
 indec. reps.

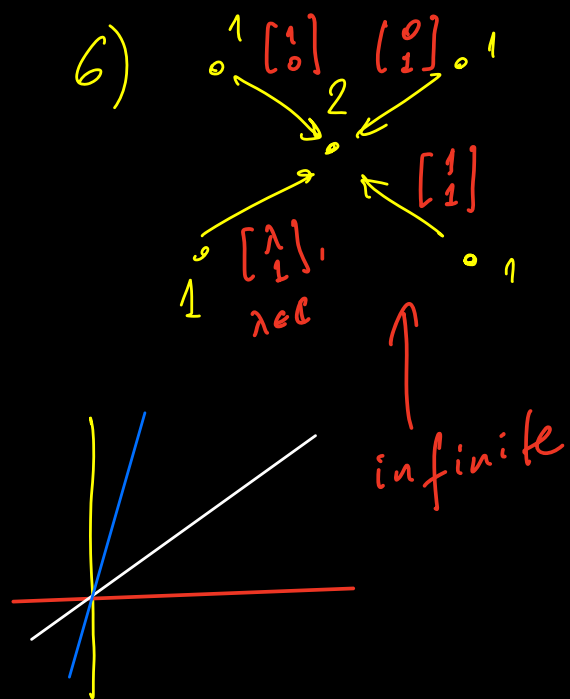


indec.

(but not irred.)



12 indec. reps



Claim: there are inf. many indec. reps
 with the dim vector $\begin{pmatrix} 1 & 1 \\ n & 2 & 1 \end{pmatrix}$

Classifying tuples of 4 pts in $\mathbb{P}^1$
 up to $PGL(2)$ -action.

The cross ratio is an invariant
 preserved by $PGL(2)$.

$\tilde{D}_4$ is of infinite type.

Thm. (Gabriel, ~1970s): A quiver Q is of finite type iff
 the underlying graph is one of the following:

A_n : $\circ - \circ - \dots - \circ$

E_6 : $\circ - \circ - \underset{\circ}{\underset{\circ}{\circ}} - \circ - \circ$

E_7 : $\circ - \circ - \underset{\circ}{\underset{\circ}{\circ}} - \circ - \circ - \circ$

D_n : $\begin{matrix} \circ \\ \diagdown \\ \circ \end{matrix} - \circ - \circ - \dots - \circ$

E_8 : $\circ - \circ - \underset{\circ}{\underset{\circ}{\circ}} - \circ - \circ - \circ - \circ$

Dynkin
 diagrams

