

$$S_d \curvearrowright V_\lambda \quad \lambda = (\lambda_1, \dots, \lambda_k) \\ \lambda_1 + \dots + \lambda_k = d$$

Lecture 12
20.10.2022

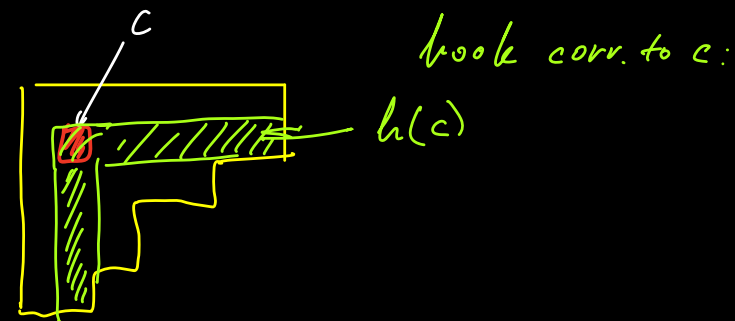
$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k \geq 1.$$

$$\begin{array}{ccc} \lambda_1 + k - 1 & & \lambda_{k-1} + 1 \quad \lambda_k \\ \text{"} & & \text{"} \\ \ell_1 & > & \dots > \ell_{k-1} > \ell_k \end{array}$$

Thm. $\dim V_\lambda = \frac{d!}{\ell_1! \dots \ell_k!} \prod_{i < j} (\ell_i - \ell_j)$

The hook length formula:

Thm. $\dim V_\lambda = \frac{d!}{\prod_{c \in \lambda} h(c)}$



Corollary. $\dim V_\lambda = \dim V_{\lambda'}$

Examples $d=5$

U

5	4	3	2	1
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 $\dim U = 1$

5	3	2	1
1			

 $\dim V = 4$

5	2	1
2		
1		

 $\dim \Lambda^2 V = 6$

4	3	1
2	1	

 $\dim W = 5$

Proof.

$l_1 - l_4$ $l_1 - l_3$ $l_1 - l_2$

9	8	5	4	2	1
6	5	2	1		
3	2				
2	1				

l_1

l_2

l_3

l_4

$$\lambda = (6, 4, 2, 2)$$

$$\ell = (9, 6, 3, 2)$$

missing:

7 6 3

4 3

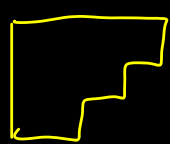
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$$\prod h(c) = \frac{l_1! l_2! \dots l_k!}{\prod_{i < j} (l_i - l_j)}$$

Exercise. We know that for $\lambda = (d), (d-1, 1)$
and their conjugates $\dim V_\lambda = 1$ and $d-1$
resp.

Find all such $\lambda \vdash d$ that
 $\dim V_\lambda < d$

Partitions and the pentagonal theorem.



$$\lambda = (\lambda_1, \dots, \lambda_k), \quad \lambda_1 + \dots + \lambda_k = n.$$

Denote # partitions by $p(n)$

Thm. $\sum_{n \geq 0} p(n) q^n = \prod_{m \geq 1} \frac{1}{1 - q^m} = 1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + \dots$

pentagonal numbers

Thm. $\prod_{m \geq 1} (1 - q^m) = 1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + q^{22} + q^{26} - \dots$

$$= 1 + \sum_{k \geq 1} (-1)^k \left[q^{\frac{k(3k-1)}{2}} + q^{\frac{k(3k+1)}{2}} \right].$$

Proof. What does the generating function

$$\prod_{m \geq 1} (1 + q^m) \quad \text{compute?}$$

$$\prod_{m \geq 1} (1 + q^m) \quad \searrow \quad a_0 + a_1 q + a_2 q^2 + \dots$$

$a_k = \#$ partitions of k
into distinct
summands

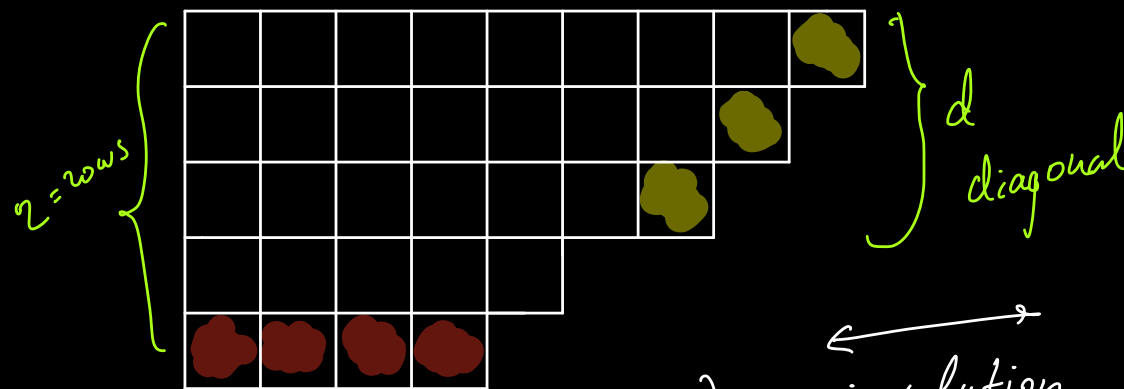
$$(1+q)(1+q^2)(1+q^3)\dots$$

$$\prod (1 - q^m) = b_0 + b_1 q + b_2 q^2 + b_3 q^3 + \dots$$

Claim: $b_k = \#$ partitions of k into an even no.
of distinct summands

— $\#$ part. into an odd number of dist.
summands.

Construct an "almost bijective" map between these
sets!

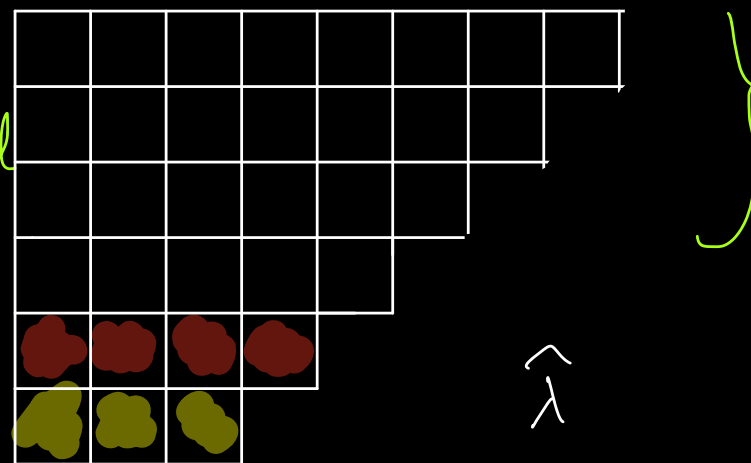


l length of last row

$d < l$:

involution

$$i^2 = Id$$

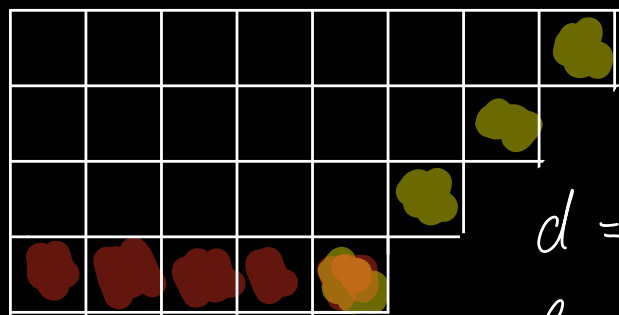


$$\hat{r} = r + 1$$

$$\hat{d} \geq d \geq \hat{l}$$

$$\hat{l} = d < l$$

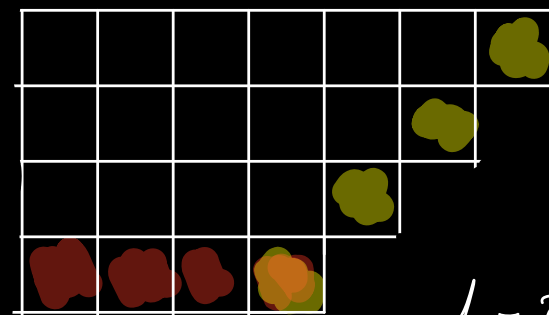
This provides a bijection of partitions into an even and odd number of distinct parts.



$$d = z$$

$$l = d + 1$$

$$z^2 + \frac{z(z+1)}{2} = \frac{z(3z+1)}{2}$$



$$d = z$$

$$l = d$$

$$z^2 + \frac{z(z-1)}{2} = \frac{z(3z-1)}{2}$$

triangular numbers

n	0	1	2	3	4	5	6	7	8	9	10	11	12
p(n)	1	1	2	3	5	7	11	15	22	30	42	56	77

$$\left(\sum_{p(0) + p(1)q + \dots} p(n) q^n \right) (1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} \dots) = 1.$$

$$p(0) = 1$$

$$p(1) - p(0) = 0$$

$$p(2) - p(1) - p(0) = 0$$

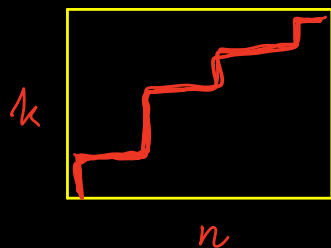
$$p(3) - p(2) - p(1) = 0$$

$$p(4) - p(3) - p(2) = 0$$

$$p(5) - p(4) - p(3) + p(0) = 0$$

$$p(n) = p(n-1) + p(n-2) - \\ - p(n-5) - p(n-7) + \\ + p(n-12) + p(n-15) - \dots$$

Gaussian binomial coefficients.



$$n \geq \lambda_1 \geq \dots \geq \lambda_k \geq 0$$

How many λ 's are there?

$$\binom{k+n}{n} = \binom{k+n}{k}.$$

$$\left[\begin{matrix} k+n \\ k \end{matrix} \right]_q = \sum_{\lambda \in k \times n} q^{|\lambda|} = 1 + q + 2q^2 + \dots + q^{kn-1} + q^{kn}$$

Example: $\left[\begin{matrix} 4 \\ 2 \end{matrix} \right] = 1 + q + 2q^2 + q^3 + q^4 = (1+q^2)(1+q+q^2).$



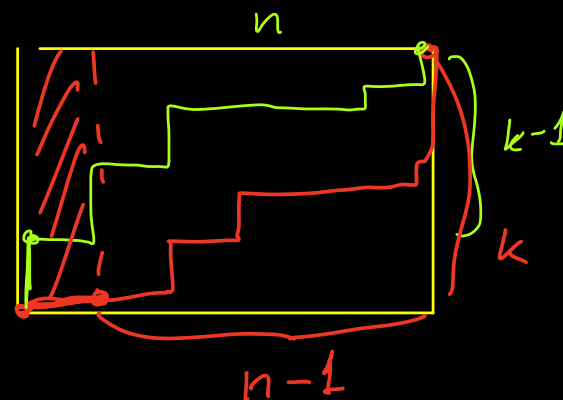
$$\left[\begin{matrix} n+1 \\ 1 \end{matrix} \right] = 1 + q + \dots + q^n, \quad \left[\begin{matrix} n \\ 0 \end{matrix} \right] = 1.$$

$$\left[\begin{matrix} k+n \\ k \end{matrix} \right]_{q=1} = \binom{k+n}{k}.$$

$$\binom{k+n}{k} = \binom{k+n-1}{k-1} + \binom{k+n-1}{k}.$$

Prop. $\left[\begin{matrix} k+n \\ k \end{matrix} \right] = \left[\begin{matrix} k+n-1 \\ k-1 \end{matrix} \right] + q^k \left[\begin{matrix} k+n-1 \\ k \end{matrix} \right]$

$$\binom{n+k}{k} = \frac{(n+k)!}{k! n!}$$



$$[n] = \begin{bmatrix} n-1 \\ 1 \end{bmatrix} = 1 + q + \dots + q^{n-1}$$

$$[n]! = [1] \cdot [2] \cdot \dots \cdot [n].$$

$$[0] = 0, \quad [1] = 1, \quad [2] = 1 + q$$

$$[0]! = 1.$$

Exercise. Show that

$$\begin{bmatrix} n+k \\ k \end{bmatrix} = \frac{[n+k]!}{[k]! [n]}.$$