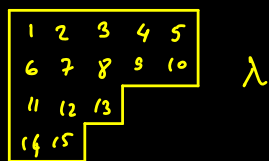


S_d

$$\lambda_1 + \dots + \lambda_k = d$$

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k \geq 1$$

 λ Partition λ

Young diagram

Lecture 11

17/10/22

$$P = P_\lambda \simeq S_{\lambda_1} \times \dots \times S_{\lambda_k} \quad \text{preserves rows}$$

$$a = a_\lambda = \sum_{g \in P_\lambda} g$$

$$Q = Q_\lambda \simeq S_{\lambda'_1} \times \dots \times S_{\lambda'_\ell} \quad \text{preserves columns}$$

$$b = b_\lambda = \sum_{g \in Q_\lambda} (\text{sgn } g) g$$

$$c = c_\lambda = a_\lambda b_\lambda \quad \text{Young symmetrizer}$$

Lemma 1 (i) If $p \in P$, $pa = ap = a$.

(ii) If $q \in Q$, $(\text{sgn } q) q b = b \cdot (\text{sgn } q) q = b$

(iii) If $p \in P$, $q \in Q$. then $p \cdot c \cdot (\text{sgn } q) q = c \quad (*)$

(iv) c is a unique element (up to a scalar) satisfying $(*)$.

Def: $\lambda > \mu$, if $\lambda_1 = \mu_1, \lambda_2 = \mu_2, \dots, \lambda_i = \mu_i, \lambda_{i+1} > \mu_{i+1}$. (i can be 0)
lexicographic ordering.

Lemma 2. (i) If $\lambda > \mu$, then $\forall x \in A = \mathbb{C}S_d$ we have $a_\lambda \times b_\mu = 0$.

In particular: if $\lambda > \mu$, $c_\lambda \cdot c_\mu = 0$.

(ii) $\forall x \in A$ $c_\lambda \cdot x \cdot c_\lambda = \text{const} \cdot c_\lambda$.

In particular, $c_\lambda^2 = n_\lambda c_\lambda, n_\lambda \in \mathbb{C}$ ($c_\lambda \sim \text{projector}$)

Proof. (i) Let $x = g \in S_d$.

$b_{\mu, gT'} = g \cdot b_{\mu, T'} \cdot g^{-1}$ is the antisymmetrizer corr. to the tableau gT' , where T' is the "canonical" tableau corr. to μ (i.e., $\begin{array}{|c|} \hline 1 & 2 & 3 & \dots & | \\ \hline \mu_1 + 1 & \dots & \dots & \dots & \\ \hline \end{array}$)

$a_{\lambda, T} \cdot b_{\mu, T'} = 0 \quad \forall T, T'$

We need to show: $a_\lambda \cdot b_\mu = 0$

"Pigeonhole principle 2.0": if $\lambda > \mu$, then there are two boxes i, j in the same row of λ and the same column of μ .

Take $t = (i \leftrightarrow j)$: $a_\lambda \cdot t = a_\lambda$
 $t b_\mu = -b_\mu \Rightarrow a_\lambda b_\mu = a_\lambda t \cdot t b_\mu = -a_\lambda b_\mu \Rightarrow a_\lambda \cdot b_\mu = 0$.

Proof of Pigeonhole Principle 2.0: Suppose the contrary.

For each row of λ , no two boxes of it belong to the same column of μ . $\Rightarrow \lambda_1 \leq \mu_1$. But $\lambda > \mu \Rightarrow \lambda_1 = \mu_1$.

Move elements of the 1st row of λ to the 1st row in μ (permuting along μ -columns). Apply induction.

(ii) $C_\lambda \times C_\lambda \sim C_\lambda$. Use Lemma 1 (iv).

$$\begin{aligned} p \cdot C_\lambda \times C_\lambda \cdot (\text{sgn } q) \cdot q &= p a_\lambda [b_\lambda \times a_\lambda] \cdot b_\lambda (\text{sgn } q) q = \\ &= a_\lambda [b_\lambda \times a_\lambda] b_\lambda (\text{sgn } q) q = C_\lambda \times C_\lambda. \quad \text{satisfies } (*). \end{aligned}$$

$$x = 1: \quad C_\lambda^2 = n_\lambda \cdot C_\lambda. \quad \square$$

$$A C_\lambda =: V_\lambda$$

Lemma 3. (i) Each V_λ is an irrep. of S_d ;

(ii) If $\lambda \neq \mu$, $V_\lambda \not\cong V_\mu$.

Proof. (i) $C_\lambda V_\lambda \subset \mathbb{C} C_\lambda$

Let $W \subset V_\lambda$ be a subrep. Then $C_\lambda \cdot W$ is 0 or $\mathbb{C} C_\lambda$

If $C_\lambda \cdot W = \mathbb{C} C_\lambda$. Then $V_\lambda = A \cdot C_\lambda \subset W$: $W = V_\lambda$.

Otherwise $C_\lambda W = 0$. $W \cdot W \subset A C_\lambda W = 0$, $W \cdot W = 0$.

Proj. $A \twoheadrightarrow W$ is given by ^{right}mult. by $\varphi \in A$, $\varphi = \varphi^2$.

But $\varphi^2 \in W \cdot W = 0 \Rightarrow \varphi = \varphi^2 = 0 \Rightarrow W = 0$.

In part, $n_\lambda \neq 0$.

Sidebar: $\mathbb{C} S_d = A = W_1^{\oplus d_1} \oplus W_2 \oplus \dots \oplus W_k^{\oplus d_k}$, W_i irred.

Want to write $\pi_i: A \rightarrow W_k$ as an element of A .

$$w_i \in W_i \quad \pi_i = \sum_{g \in G} e_g w_i$$

$$1_A = \pi_1 \oplus \pi_2 \oplus \dots \oplus \pi_k.$$

$$A = \bigoplus_{\downarrow} \text{End}(d_i) \\ (0, \dots, \text{Id}_i, \dots)$$

If $W \subset V_\lambda$, $W = A \cdot w$, $w = x \cdot c_\lambda$. ?

(ii) Suppose $\lambda > \mu$. $c_\lambda V_\lambda = \mathbb{C} \cdot c_\lambda \neq 0$.

$c_\lambda \cdot V_\mu = c_\lambda \cdot A c_\mu = 0$, so $V_\lambda \neq V_\mu$. $\square$

Lemma 4. For each λ we have $c_\lambda \cdot c_\lambda = n_\lambda \cdot c_\lambda$,

where $n_\lambda = \frac{d!}{\dim V_\lambda}$.

Proof. Let $F: A \rightarrow A$
 $x \mapsto x c_\lambda$, $\text{Im } F = V_\lambda$.

$F|_{V_\lambda} = n_\lambda \cdot \text{Id}$

$\text{Tr } F = \dim V_\lambda \cdot n_\lambda$.

$\text{Tr } F = \sum_{g \in S_d} \left(\begin{array}{c} \text{coef. in front of } e_g \\ \text{in } e_g \cdot c_\lambda \end{array} \right) = \sum_{g \in S_d} 1 = d! \quad \square$

The Frobenius character formula.

$$\chi_{V_\lambda}(C_{\underline{i}}) = ?$$

Let $\underline{i} = (i_1, \dots, i_d)$, $\sum \alpha \cdot i_\alpha = d$.

$C_{\underline{i}}$ conj. class with i_1 stable pts, i_2 transp., ...
 i_m m -cycles ...

$$\lambda = (\lambda_1, \dots, \lambda_k)$$

$x_1, \dots, x_k$ variables

$$P_j(x) = x_1^j + \dots + x_k^j.$$

$$\Delta(x) = \prod_{i < j} (x_i - x_j) = \det(x_i^{k+1-j})_{i,j=1}^k$$

Vandermonde det.

$f(x)$ power series, $(\ell_1, \dots, \ell_k)$ multi-index

$[f(x)]_{(\ell_1, \dots, \ell_k)}$ = the coef. in front of $x_1^{\ell_1} x_2^{\ell_2} \dots x_k^{\ell_k}$.

If $\lambda_1 \geq \dots \geq \lambda_k$ part., let $\ell_1 = \lambda_1 + k - 1, \ell_2 = \lambda_2 + k - 2, \dots, \ell_k = \lambda_k$. $\ell_1 > \dots > \ell_k$.

Thm. (Frobenius)

$$\chi_{\lambda}(C_{\underline{i}}) = \left[\Delta(x) \cdot \prod_j p_j(x)^{i_j} \right]_{(l_1, \dots, l_k)}$$

Example: $d=5$, $\lambda = (3, 2)$, $C_i \ni (12) \cdot (345)$
 $\ell = (4, 2)$

$$i_1 = 0, \quad i_2 = 1, \quad i_3 = 1$$

$$\chi_{(3,2)}(C_{\underline{i}}) = \left[(x_1 - x_2)(x_1^2 + x_2^2)(x_1^3 + x_2^3) \right]_{(4,2)} = 1.$$

Equivalent form in terms of Schur polynomials:

$$\prod_j p_j(x)^{i_j} = \sum \chi_{\lambda}(C_{\underline{i}}) s_{\lambda}, \quad s_{\lambda} - \text{Schur polynomials.}$$

Find $\dim V_{\lambda} = \chi_{\lambda}(e) !$

$$\dim V_{\lambda} = \chi_{\lambda}(e) = \left[\Delta(x) \cdot (x_1 + \dots + x_k)^d \right]_{(l_1, \dots, l_k)}.$$

$$\Delta(x) = \begin{vmatrix} 1 & x_k & \dots & x_k^{k-1} \\ \vdots & \vdots & & \vdots \\ 1 & x_1 & \dots & x_1^{k-1} \end{vmatrix} = \sum_{\sigma \in S_k} (\text{sgn } \sigma) x_k^{\sigma(1)-1} \dots x_1^{\sigma(k)-1}$$

$$(x_1 + \dots + x_k)^d = \sum \frac{d!}{r_1! \dots r_k!} x_1^{r_1} \dots x_k^{r_k}, \quad r_1 + \dots + r_k = d.$$

Coef. in front of $(l_1 \dots l_k)$ is

$$\sum (\text{sgn } \sigma) \frac{d!}{(l_1 - \sigma(k)+1)! \dots (l_k - \sigma(1)+1)!}, \quad \text{sum is taken for all } \sigma \text{ s.t.}$$

||

$$\frac{d!}{l_1! \dots l_k!} \sum_{\sigma \in S_k} (\text{sgn } \sigma) \prod_{j=1}^k \overset{\substack{\ell_j \\ \parallel}}{\ell_j} (\ell_j - 1) \dots (\ell_j - \overset{\downarrow \sigma(k-j+1)+1}{\sigma(k-j+1)+1} + 1) =$$

$$= \frac{d!}{l_1! \dots l_k!} \begin{vmatrix} 1 & l_k & l_k(l_k-1) & \dots \\ \vdots & \vdots & \vdots & \\ 1 & l_1 & l_1(l_1-1) & \dots \end{vmatrix} =$$

$$= \frac{d!}{l_1! \dots l_k!} \prod_{i < j} (l_i - l_j).$$

Thm. $\dim V_\lambda = \frac{d!}{l_1! \dots l_k!} \prod_{i < j} (l_i - l_j), \quad l_i = \lambda_i + k - i.$

hook length formula.