

Representations of S_d

Lecture 10

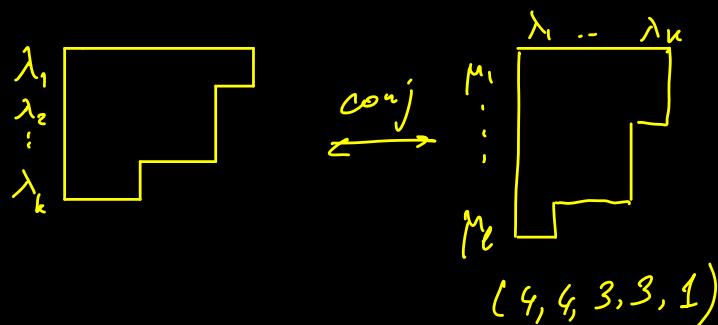
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Conj. classes: indexed by partitions

Partition: $d = \lambda_1 + \lambda_2 + \dots + \lambda_k$, $\lambda_i \geq 1$,
 $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots \geq \lambda_k$

Young diagram:

$$5 + 4 + 4 + 2 = 15$$



$$\mu = (\mu_1, \dots, \mu_e)$$

"
 λ' conjugate partition

d	0	1	2	3	4	5	6	...
$p(d)$	1	1	2	3	5	7	11	...

Bad news: no closed formula for $p(d)$!

Thm. (Euler): $P(q) = \sum_{d \geq 0} p(d) \cdot q^d = \prod_{n \geq 1} \frac{1}{1 - q^n}$.

Sketch of proof. $\frac{1}{1 - q^n} = 1 + q^n + q^{2n} + \dots$

$$\prod \frac{1}{1 - q^n} = (1 + q + q^2 + \dots)(1 + q^2 + q^4 + \dots)(1 + q^3 + q^6 + \dots) \dots$$

Question: How to compute $\prod_{n \geq 1} (1 - q^n)$?

Answer:

$$\prod_{n \geq 0} (1 - q^n) = 1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + q^{22} + q^{26} - q^{35} - q^{40} + \dots$$

★ The coef. are 0 or ± 1 .

★ $-, -, +, +, -, -, +, +,$

★ The differences in each pair are 1, 2, 3, 4, ...

★ What is $0, 1, 5, 12, 22, 35, \dots$?
1 4 7 10 13

The pentagonal numbers: $\frac{n(3n-1)}{2}$.

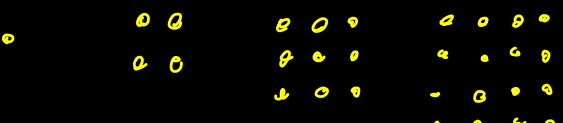
Powers of q are:

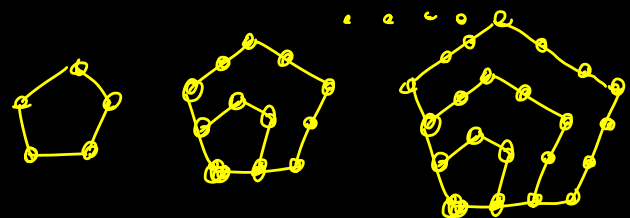
$$\frac{n(3n-1)}{2} \text{ on}$$

$$\frac{n(3n-1)}{2} + n =$$

$$= \frac{n(3n+1)}{2}.$$

 1, 3, 6, 10... triangles
 $\binom{n}{2}$

 1, 4, 9, 16... squares

 pentagons

1 5 12 22

Thm. (Euler): $\prod_{n \geq 1} (1 - q^n) = 1 + \sum_{n \geq 1} (-1)^n \left[q^{\frac{n(3n-1)}{2}} + q^{\frac{n(3n+1)}{2}} \right]$

Thm. Asymptotically $p(d) \sim \frac{1}{\alpha d} e^{\beta \sqrt{d}}$, $\alpha = 4\sqrt{3}$
 $\beta = \pi \sqrt{\frac{2}{3}}.$

Prop: Partitions $\lambda = (\lambda_1, \dots, \lambda_k)$ of d bij. correspond to the conj classes of S_d : $\lambda \longleftrightarrow \text{perm. with cycle lengths } \lambda_1, \lambda_2, \dots, \lambda_k.$

Tableau
(pl. tableaux) $\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 7 & 8 & \\ \hline 9 & & \\ \hline \end{array} T$

a filling of boxes of λ with integers
 $1, \dots, d$

Take two subgroups: $P = P_\lambda = \{g \in S_d \mid g \text{ preserves rows of } T\}.$

$Q = Q_\lambda = \{g \in S_d \mid g \text{ preserves columns of } T\}.$

$$P \simeq S_{\lambda_1} \times S_{\lambda_2} \times \dots \times S_{\lambda_k}$$

$$Q \simeq S_{\lambda'_1} \times S_{\lambda'_2} \times \dots \times S_{\lambda'_\ell}.$$

Notation: $A = \mathbb{C} S_d = \left\{ \sum_{g \in S_d} c_g e_g \mid c_g \in \mathbb{C} \right\}.$
 $e_g \cdot e_h = e_{gh}.$

$$a_\lambda = \sum_{g \in P} e_g$$

row symmetrizer

$$b_\lambda = \sum_{g \in Q} (\text{sgn } g) \cdot e_g$$

column antisymmetrizer

$$\mathbb{C} S_d \ni c_\lambda = a_\lambda \cdot b_\lambda.$$

Young symmetrizer.

Examples: 1) $\lambda = (d)$, $\boxed{}\boxed{}\boxed{}\boxed{}\boxed{}$ $a_\lambda = \sum_{g \in S_d} e_g$

$$b_\lambda = 1$$

$$A \rightarrow A$$

$$x \mapsto x \cdot C_d$$

$\sim$ projector to the trivial subrep U

$$C(d) = a_{(d)} = \sum_{g \in S_d} e_g : \text{projector to the space of invariants}$$

2) $\lambda = (1, \dots, 1)$: $\begin{array}{|c|} \hline \\ \hline \\ \hline \\ \hline \end{array}$

$$a_\lambda = 1$$

$$b_\lambda = \sum (\text{sgn } g) e_g$$

$$A \rightarrow A$$

$$x \mapsto x \cdot c_{(1, \dots, 1)} \sim \text{projector onto } U'$$

3) $d=3$, $\lambda = (2, 1)$

$$\boxed{\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array}}$$

$$a_\lambda = (e_1 + e_{(12)})$$

$$b_\lambda = (e_1 - e_{(13)})$$

$$C_{(2,1)}$$

$$C_\lambda = (e_1 + e_{(12)})(e_1 - e_{(13)}) = e_1 + e_{(12)} - e_{(13)} -$$

$$A \cdot C_{(12)} = \langle C_{(2,1)}, (13)C_{(2,1)} \rangle \simeq V$$

Let V be a vector space

$S_d \curvearrowright V^{\otimes d} = V \otimes \dots \otimes V$ by permuting components

$$\mathbb{C} S_d \curvearrowright V^{\otimes d}$$

$\downarrow$

a_λ, b_λ

$$a_\lambda: V^{\otimes d} \rightarrow V^{\otimes d}$$

$$v_1 \otimes \dots \otimes v_d \mapsto \dots$$

$$\text{Im } a_\lambda = \text{Sym}^{\lambda_1} V \otimes \text{Sym}^{\lambda_2} V \otimes \dots \otimes \text{Sym}^{\lambda_k} V$$

$$\text{Im } b_\lambda = \wedge^{\lambda'_1} V \otimes \wedge^{\lambda'_2} V \otimes \dots \otimes \wedge^{\lambda'_m} V.$$

Thm. (a) C_λ is a multiple of an idempotent:

$$C_\lambda^2 = n_\lambda C_\lambda.$$

$\swarrow$ left ideal in A

$$A \twoheadrightarrow A C_\lambda \subset A$$

$$x \mapsto x C_\lambda$$

(b) $A C_\lambda \cong V_\lambda$ is irreducible;

(c) All irreps of S_d are obtained in this way.

Proof. $P = P_\lambda, Q_\lambda = Q \subset S_d$. $P \cap Q = \{1\}$, so each $g \in S_d$ is presented as $g = p \cdot q$, $p \in P, q \in Q$ not more than uniquely.

$$c_\lambda = a_\lambda \cdot b_\lambda = \left(\sum_{p \in P} e_p \right) \left(\sum_{q \in Q} (\text{sgn } q) e_q \right) = \sum \pm e_g,$$

s.t. $g \in P \cdot Q$, $\pm 1 = \text{sgn } q$.
In part., e_1 comes with a +.

Lemma. (i) If $p \in P$, then $p \cdot a = a \cdot p = a$

(ii) If $q \in Q$, then $(\text{sgn } q) q \cdot b = b \cdot (\text{sgn } q) q = b$.

(iii) If $p \in P, q \in Q$, then $p \cdot c \cdot (\text{sgn } q) q = c$, (*)

(iv) c is a unique element with such a property, up to a scalar multiple.

Proof. (i), (ii) obvious. (iii) $\Leftarrow$ (i) + (ii):

$$p \cdot c \cdot (\text{sgn } q) q = p \cdot a \cdot b \cdot (\text{sgn } q) q = a \cdot b = c.$$

(iv) let $\sum n_g e_g$ satisfy (*).

Then $n_{pgq} = (\text{sgn } q) n_g$. In part.,
 $n_{pq} = (\text{sgn } q) n_1$.

Need to show: $n_g = 0$ for $g \notin P \cdot Q$.

For such a g , we find a transp. t , s.t. ($t^2 = 1$)

$p = t \in P$, $q = g^{-1} t g \in Q$. So

$pgq = g$ $n_g = (\text{sgn } q) n_g$, $n_g = -n_g$

T tableau

$\downarrow$

$T' = gT$:

number i replaced
by $g(i)$.

Claim: there exist two entries

α, β in one row of T
in one column of T' .

$t = (\alpha \leftrightarrow \beta)$.

Need to show: if there is no such pair (α, β) , then

$g = p \cdot q$, for some $p \in P$, $q \in Q$.

$p_i \in P$, $q'_i \in Q' = g Q g^{-1}$, s.t. the tableaux $p_i T$ and $q'_i T'$ have the same 1st row:

1	2	3	4
5	6	7	
8			

T

7	5	6	4
8	3	2	
1			

T'

1	3	2	4
5	6	7	
8			

1	3	2	4
7	5	6	
8			

At the end :

$p \in P$, $q' \in Q'$: s.t.

$$p T = q' T'$$

$$p T = q' g T \quad p = q' g,$$

$$g = p \cdot (q')^{-1} \in P \cdot Q.$$