

Last time : rep. of S_4

Lecture 7

26/09/22

Reps:

1	U	trivial
1	U'	sign rep: $\sigma \mapsto \text{sgn.}$
3	V	tautological: $\mathbb{C}^4 \simeq U \oplus V$ perm.rep
3	$V' = V \otimes U'$	
2	W	$S_4/V_4 \simeq S_3 \curvearrowright \mathbb{C}^2$

Alternating group (even permutations) $A_4 \subset S_4$.

$$A_n \triangleleft S_n$$

$$|A_n| = n!/2$$

Conj. classes of S_4 :

$e +$ $(12) -$ $(123) +$ $(12)(34) +$ $(1234) -$

3 conj. classes? NO!

Warning! Some conj classes may split:

σ conj to σ' in S_n , but not in A_n .

$$S_3 \supset A_3 = \{e, (123), (132)\}.$$

$\nwarrow \nearrow$
conj in S_3 , but not in A_3 .

Conj classes of A_4 :

$$\{e\}, \quad \{(12)(34), (13)(24), (14)(23)\},$$

The set of all 3-cycles is not a conj. class:

there are 8 of them. $|A_4| = 12$,

$$8 \nmid 12.$$

Exercise. Prove that there are two conj. classes of 3-cycles, containing (123) and (213) esp.

General question: what happens with a rep. V of a group G while we restrict to a subgroup $H \subset G$?

$G \rightarrow GL(V)$, say, V is irred.

$\begin{matrix} V \\ H \end{matrix} \nearrow$

V is an H -rep,
but not necessarily irred.

[the converse is true: V H -irrep $\Rightarrow V$ G -irrep].

$$\chi_{V,H}(h) = \chi_{V,G}(h)$$

A_4	e 1	(123) 4	(132) 4	$(12)(34)$ 3	
U	1	1	1	1	trivial.
V	3	0	0	-1	
W'	1	$x \omega$	$y \omega^2$	z	1
W''	1	ω^2	ω	1	

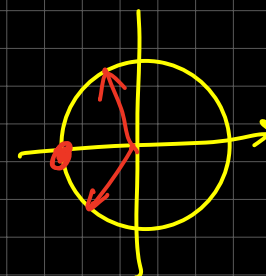
$$\langle x_{W'}, x_V \rangle = 3 + 3 \cdot (-1)z = 0 \quad z = 1. \quad \omega = \sqrt[3]{1}$$

$$\langle x_{W'}, x_U \rangle = 1 + 4(x+y) + 3 = 0 \quad x+y = -1$$

$$\langle x_{W'}, x_{W'} \rangle = 1 + 4|x|^2 + 4|y|^2 + 3 = 12 \quad |x|^2 + |y|^2 = 2$$

$$x, y = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i.$$

$$\operatorname{Re} x = \operatorname{Re} y = -\frac{1}{2}$$



$$\chi_w: \quad 2 \quad -1 \quad -1 \quad 2$$

$$\langle \chi_w, \chi_w \rangle = \frac{1}{12} (2^2 + 4 + 4 + 3 \cdot 4) = 2.$$

How to compute $\chi_{w'}$ and $\chi_{w''}$?

What are these 1-dim'l reps?

U' restricted to A_4 is $\simeq U$

Is V irred. as an A_4 -rep?

$$\langle \chi_V, \chi_V \rangle = \frac{1}{12} (3^2 + 3 \cdot (-1)^3) = 1.$$

How to do this without computations?

$$A_4 \subset S_4 \twoheadrightarrow S_3 \supset A_3 \simeq \mathbb{Z}/3\mathbb{Z}$$

Next goal: S_5 and A_5 .

S_5	e	(12) 10	(123) 20	(1234) 30	(12345) 24	$(12)(34)$ 15	$(12)(345)$ 20
triv. U	1	1	1	1	1	1	1
sign U'	1	-1	1	-1	1	1	-1
tautolog. V	4	2	1	0	-1	0	-1
$V' = V \circ U$	4	-2	1	0	-1	0	1
$\Lambda^2 V$	6	0	0	0	1	-2	0
W	5	α_1 1	α_2 -1	α_3 -1	α_4 0	α_5 1	α_6 -1
W'	5	$-\alpha_1$ -1	α_2 -1	$-\alpha_3$ 1	α_4 0	α_5 1	$-\alpha_6$ 1

$$\chi_{\wedge^2 V}(g) = \frac{1}{2} (\chi_V(g)^2 - \chi_V(g^2))$$

$$\chi_{\wedge^2 V}((12)) = \frac{1}{2} (4 - 4) = 0$$

$$120 = 1^2 + 1^2 + 4^2 + 4^2 + n_1^2 + n_2^2 + n_3^2$$

$$120 - 34 = 86 = n_1^2 + n_2^2 + n_3^2$$

$$\chi_{\wedge^2(V)}^{(12345)} =$$

$$= \frac{1}{2} ((-1)^2 - (-1)) = 1$$

$$5^2 + 5^2 + 6^2$$

$$1^2 + 7^2 + 6^2$$



There is no other one-dim'l rep U'' : otherwise

$$U'' \otimes U' \simeq U'', \quad \text{nonsense.}$$

Another way:

ρ : 1-dim'l:

$$G \longrightarrow \mathbb{C}^*$$

$$\rho(xy x^{-1} y^{-1}) = 1$$

$$(S_5)' = A_5$$

$$\rho: [G, G] \longrightarrow 1$$

$$1\text{-dim'l reps} = \text{reps of } G/[G, G] = S_5/A_5 = \mathbb{Z}/2\mathbb{Z}$$

$$\langle x_u, x_w \rangle = 5 + 10\alpha_1 + 20\alpha_2 + 30\alpha_3 + 24\alpha_4 + 15\alpha_5 + 20\alpha_6 = 0$$

$$\langle x_{u'}, x_w \rangle = 5 - 10\alpha_1 + 20\alpha_2 - 30\alpha_3 + 24\alpha_4 + 15\alpha_5 - 20\alpha_6 = 0$$

$$\langle x_v, x_w \rangle = 20 + 20\alpha_1 + 20\alpha_2 - 24\alpha_4 - 20\alpha_6 = 0$$

$$\langle x_{v'}, x_w \rangle = 20 - 20\alpha_1 + 20\alpha_2 - 24\alpha_4 + 20\alpha_6 = 0$$

$$\langle x_{uv}, x_w \rangle = 30 + 24\alpha_4 - 30\alpha_5 = 0$$

$$\alpha_1 = \alpha_6 = -\alpha_3$$

$$40 + 40\alpha_2 - 48\alpha_4 = 0$$

$$40\alpha_1 = 40\alpha_6$$

$$U \oplus V \oplus W$$

$$5 + 5\alpha_2 - 6\alpha_4 = 0$$

Exercise: decompose $S^2 V$, find W inside.

A_5	e		(123) 20	(21345) 12	(12345) 12	$(12)(34)$ 15	
triv. U	1		1	1	1	1	
tautolog. V	4		1	-1	-1	0	
$\Lambda^2 V$	6		0	1	1	-2	not irr.
W	5		-1	0	0	1	irred.
X	3						
Y	3						

What reps of A_5 are isom?

$$U \cong U', \quad V \cong V', \quad W \cong W'$$

$$\langle \chi_W, \chi_W \rangle = (25 + 20 + 15) \frac{1}{60} = 1.$$

$$\langle \chi_{\wedge^2 V}, \chi_{\wedge^2 V} \rangle = (36 + 12 + 12 + 60) = 2.$$

$$60 = 1^2 + 4^2 + 5^2 + m^2 + n^2$$

$$18 = m^2 + n^2$$

Homework: compute χ_X, χ_Y .