

Lecture 6

22/09/22

G finite group,

$$\rho: G \rightarrow GL(V), \quad V \cong \mathbb{C}^n.$$

Character: $\chi_\rho: G \rightarrow \mathbb{C}$

$$g \mapsto \text{tr } \rho(g).$$

$$\mathbb{C}^{\text{class}}_G = \{f: G \rightarrow \mathbb{C} \mid f(g) = f(hgh^{-1}) \forall h\}$$

χ_ρ

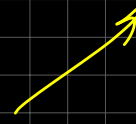
$$\langle \alpha, \beta \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\alpha(g)} \cdot \beta(g), \quad \leftarrow \text{Hermitian inner product on } \mathbb{C}^{\text{class}}_G.$$

Thm. $\{\chi_\rho \mid \rho \text{ irrep of } G\}$ form an orthonormal basis of $\mathbb{C}^{\text{class}}_G$.

$$\langle \chi_v, \chi_w \rangle = \begin{cases} 1, & v \cong w \\ 0, & v \not\cong w. \end{cases}$$

Cor. $\# \text{ irreps of } G = \# \text{ conj. classes of } G.$

Thm. $R = \bigoplus V_i^{d_i}$, $d_i = \dim$ of V_i . V_i 's are the irreps of G .

Cor. $|G| = \sum d_i^2$. 

Some simple examples.

$G = S_3$. permutations of $\{1, 2, 3\}$. $3! = 6$

Conj. classes: e , $\left\{ \begin{matrix} (12) \\ (13) \\ (23) \end{matrix} \right\}$, $\left\{ \begin{matrix} (123) \\ (132) \end{matrix} \right\}$.

The character table

	conj class g
V irrep	$\chi_V(g)$

S_3	e	(12)	(123)
	1	3	2
U	1	1	1
U'	1	-1	1
V	$x = 2$	$y = 0$	$z = -1$

$$1^2 + 1^2 + x^2 = 6$$

triv. rep

sign rep

$$x = \dim V.$$

Suppose we don't know anything about V !

$$\langle x_u, x_v \rangle = 0$$

$$\langle x_{u'}, x_v \rangle = 0$$

$$\langle x_v, x_v \rangle = 1.$$

$$\begin{cases} x + 3y + 2z = 0 \\ x - 3y + 2z = 0 \\ \bar{x}x + 3\bar{y}y + 2\bar{z}z = 6 \end{cases}$$

$$\begin{cases} y = 0 \\ x + 2z = 0 \\ |x|^2 + 2|z|^2 = 6 \end{cases}$$

$$\Leftrightarrow \begin{cases} x = \pm 2 & x > 0 ! \\ y = 0 \\ z = \mp 1 \end{cases}$$

S_4	e	(12)	(123)	(1234)	$(12)(34)$
	1	6	8	6	3
U	1	1	1	1	1
U'	1	-1	1	-1	1
V	3	1	0	-1	-1
V'	3	-1	0	1	-1
W	2	0	-1	0	2

$$|S_4| = 4! = 24$$

trivial

sign

tautological rep.

$$24 = 1^2 + 1^2 + 3^2 + d_1^2 + d_2^2$$

$$d_1^2 + d_2^2 = 13$$

$$4^2 + 6 \cdot 2^2 + 8 \cdot 1$$

$$\text{Perm} \quad 4 \quad 2 \quad 1 \quad 0 \quad 0$$

$$\langle \chi_{\text{perm}}, \chi_{\text{perm}} \rangle = 2$$

$$\text{Perm} = U \oplus V$$

$$G \curvearrowright X$$

$x \in X$ is g -stable if $g \cdot x = x$.

$$G \rightarrow GL(V), \quad V = \langle e_x \rangle.$$

$$\chi(V \otimes V) = (9, 1, 0, 1, 1)$$

$$\text{mult}(U; V \otimes V) = \langle \chi_U, \chi_{V \otimes V} \rangle = \frac{1}{24} (9 + 6 + 6 + 3) = 1.$$

$$\text{mult}(U'; V \otimes V) = \langle \chi_{U'}, \chi_{V \otimes V} \rangle = \frac{1}{24} (9 - 6 - 6 + 3) = 0.$$

$$\text{mult}(V; V \otimes V) = \langle \chi_V, \chi_{V \otimes V} \rangle = \frac{1}{24} (27 + 6 - 6 - 3) = 1$$

$$\text{mult}(V'; V \otimes V) = \frac{1}{24} (27 - 6 + 6 - 3) = 1.$$

$$V \otimes V \cong U \oplus V \oplus V \oplus W$$

$$9 = 1 + 3 + 3 + 2$$

$$R = U \oplus U' \oplus 3 \cdot V \oplus 3 \cdot V' \oplus 2 \cdot W$$

$$\chi_R = (24, 0, 0, 0, 0)$$

$$\chi_W = \frac{1}{2} (\chi_R - \chi_U - \chi_{U'} - 3\chi_V - 3\chi_{V'})$$

$$\begin{cases} 2 + 6x_1 + 8x_2 + 6x_3 + 3x_4 = 0 \\ 2 - 6x_1 + 8x_2 - 6x_3 + 3x_4 = 0 \\ 6 + 6x_1 - 6x_3 - \overset{3}{\cancel{6}}x_4 = 0 \\ 6 - 6x_1 + 6x_3 - \overset{3}{\cancel{6}}x_4 = 0 \end{cases}$$

Question: What is this 2-dim'l rep. W ?

Where does it come from?

Question: What makes S_4 special?

V_4 - the Kleinian four-group.

$$\{e, (12)(34), (13)(24), (14)(23)\} \trianglelefteq S_4$$

Fact: $S_4/V_4 \xrightarrow{\sim} S_3$.

Explanation 1: $S_4 \curvearrowright \mathbb{C}[x_1, x_2, x_3, x_4]$.

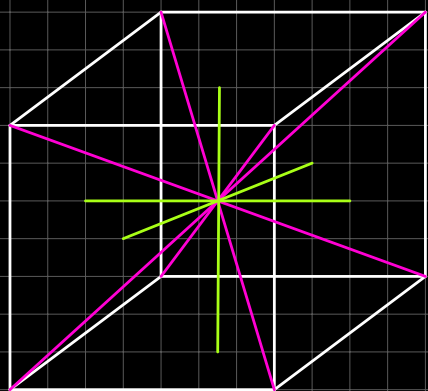
$$x_1 x_2 + x_3 x_4 \quad x_1 x_3 + x_2 x_4 \quad x_1 x_4 + x_2 x_3$$

S_4 acts on a 3-element set (of these polynomials)

Kernel of this action is V_3 .

$$S_4 / V_4 \xrightarrow{\sim} S_3$$

Explanation 2: Rotations of a cube $\cong S_4$



24 rotations of a cube

4 big diagonals

$$\text{Rot}(C) \xrightarrow{\sim} S_4 = \text{perm. of big diag.}$$

Exercise: this is an iso.

$$\text{Rot}(C) \xrightarrow{\pi} S_3 = \text{"axes", or pairs of opposite faces.}$$

$|S$

S_4

$$\text{Ker } \pi = V_4. \quad (\text{Exercise})$$

Now for a rep. $\rho: S_3 \rightarrow GL(W)$,

we have a rep $\tilde{\rho}: S_4 \twoheadrightarrow S_3 \rightarrow GL(W)$

$\text{Ker } \tilde{\rho}$ is V_4 .

Mult table for reps.

(x)

S_4	e 1	(12) 6	(123) 8	(1234) 6	$(12)(34)$ 3
u	1	1	1	1	1
u'	1	-1	1	-1	1
v	3	1	0	-1	-1
v'	3	-1	0	1	-1
w	2	0	-1	0	2

$$w \otimes u' = w$$

$$v \otimes v = u \oplus v \oplus v' \oplus w$$

$$v \otimes v' = (v \otimes v) \otimes u' =$$

$$= (u \oplus v \oplus v' \oplus w) \otimes u' =$$

u	u'	v	v'	w
u'	u	v'	v	w
v	v'	$u \oplus v \oplus v' \oplus w$	$u' \oplus v \oplus v' \oplus w$	?
v'	v	$u' \oplus v \oplus v' \oplus w$	$u \oplus v \oplus v' \oplus w$	?
w	w			?

Homework:

① Complete this table:

find $v \otimes w$,

$v' \otimes w$, $w \otimes w$.

(2) $S_4 \cong \text{Rot}(\text{Cube})$.

S_4 acts on functions on the

- vertices 8
- edges 12
- faces of C. 6

Decompose these reps (of dim 8, 12, 6) into
irreps.