

"Advanced linear algebra, con'd"

Lecture 4

noncomm. mult. 15/09/22

$V \otimes W$ tensor product

$$v \otimes w, \quad v \in V, \quad w \in W$$

$S^k V$ symmetric powers

$$v_1 \cdot v_2 \cdots v_k = v_{\sigma(1)} \cdots v_{\sigma(k)}$$

commutative

$$\forall \sigma \in S_k.$$

Exterior powers of a vector space
(=wedge powers).

Def. Let V, W be two vector spaces, $k \in \mathbb{N}$ char $K \neq 2$

A k -linear map

$$V \times \cdots \times V \longrightarrow W$$

is said to be antisymmetric (=skew-symmetric)

$$\text{if } \varphi(v_1, \dots, v_i, v_{i+1}, \dots, v_k) = -\varphi(v_1, \dots, v_{i+1}, v_i, \dots, v_k).$$

$$\varphi(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = (-1)^{\sigma} \varphi(v_1, \dots, v_k).$$

Def. A vector space Λ together with an antisym. k -linear map

$$V \times \dots \times V \longrightarrow \Lambda$$

$$v_1, \dots, v_k \longmapsto v_1 \wedge v_2 \wedge \dots \wedge v_k,$$

is called the k -th exterior power of V , if

for a certain basis $e_1, \dots, e_n$ of V , the elements $e_{i_1} \wedge \dots \wedge e_{i_k}$, $1 \leq i_1 < i_2 < \dots < i_k \leq n$, form a basis of Λ .

Prop. Such a space exists and does not depend upon a choice of $e_1, \dots, e_n$.

Pf. Existence: define $\Lambda = \text{Span} \{ f_{i_1, \dots, i_k} \}$
 $\wedge: (e_{i_1}, \dots, e_{i_k}) \mapsto f_{i_1, \dots, i_k}.$

Indep. upon a choice of basis: $e'_1, \dots, e'_n$ another basis

$e_{i_1} \wedge \dots \wedge e_{i_k}$ can be expressed linearly via $e_{i_1} \wedge \dots \wedge e_{i_k}$ and vice versa.

Exercise: Formulate and prove the uniqueness statement.

Notation: $\Lambda^k V := \Lambda$

Prop: $\dim \Lambda^k V = C_n^k = \binom{n}{k} = \frac{n!}{k!(n-k)!}$.

Pf: the basis is given by $e_{i_1} \wedge \dots \wedge e_{i_k}$, $1 \leq i_1 < \dots < i_k \leq n$.

$\dim S^k V = \binom{n+k-1}{k}$. # ways to put k apples into n boxes.
 $\dim \Lambda^k V = 0$ for $k > n$, $\dim \Lambda^n V = 1$.

Let $A: V \rightarrow V$ be a linear operator.

$$(\Lambda^k A)(v_1 \wedge \dots \wedge v_k) = Av_1 \wedge \dots \wedge Av_k.$$

$$(\wedge^n A)(v_1 \wedge \dots \wedge v_n) = \lambda_A \cdot (v_1 \wedge \dots \wedge v_n) \quad (*)$$

$$\lambda_A = \det A.$$

Exercise. Show that this is equivalent to the formula

$$\det A = \sum_{\sigma \in S_n} (-1)^\sigma a_{1\sigma(1)} \dots a_{n\sigma(n)}.$$

Exercise Obtain the formulas for the row and column expansion of $\det$.

$$\wedge^k A: V \longrightarrow V$$

$$\operatorname{tr} \wedge^k A = ?$$

Exercise. Show that $\operatorname{tr} \wedge^k A$ is equal to

$\pm$ coeff. in front of λ^{n-k} in $\chi_A(\lambda) = \det(A - \lambda I)$.

$$\det(A \otimes B) =$$

$$A: V \rightarrow V$$

matrices A

w.r.t. $e_1, \dots, e_n$

$$B: W \rightarrow W$$

B

w.r.t. $f_1, \dots, f_m$

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}B & a_{n2}B & \dots & a_{nn}B \end{pmatrix}$$

Suppose that A, B are upper triangular? (we can obtain this by a change of bases $e_1, \dots, e_n, f_1, \dots, f_m$)

$$A \otimes B = \begin{pmatrix} \boxed{a_{11} B} & a_{12} B & \dots & a_{1n} B \\ 0 & \boxed{a_{22} B} & \dots & a_{2n} B \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \boxed{a_{nn} B} \end{pmatrix}$$

$$A = \begin{pmatrix} \lambda_1 & * \\ 0 & \ddots & \lambda_n \end{pmatrix}$$

$$B = \begin{pmatrix} \mu_1 & * \\ & \ddots & \mu_m \end{pmatrix}$$

(//)

$$a_{ii} B = \begin{pmatrix} \lambda_i \mu_1 & & & \\ 0 & \lambda_i \mu_2 & & \\ 0 & 0 & \ddots & \\ 0 & \dots & \dots & 0 \end{pmatrix}$$

$A \otimes B$ is upper triangular with the diag entries $\lambda_i \mu_j$.

$$\text{tr}(A \otimes B) = \sum_{\substack{i=1 \dots n \\ j=1 \dots m}} \lambda_i \mu_j = (\lambda_1 + \dots + \lambda_n) \cdot (\mu_1 + \dots + \mu_m) = \text{tr} A \cdot \text{tr} B.$$

$$\det(A \otimes B) = \prod_{i=1}^n \lambda_i^m \mu_1 \mu_2 \dots \mu_m =$$

$$= (\lambda_1 \dots \lambda_n)^m \cdot (\mu_1 \dots \mu_m)^n = (\det A)^m (\det B)^n.$$

Recall. If A, B are diag. with eigenvalues $\lambda_1, \dots, \lambda_n, \mu_1, \dots, \mu_m$, then $A \otimes B$ is diag. with eigenval. $\lambda_i \mu_j$.

Over $\mathbb{C}$, use the trick: diagonalizable matrices form an open dense subset of $\text{Mat}_n(\mathbb{C})$.

Let us compute $\text{tr } S^k A$ and $\text{tr } \Lambda^k A$:

again, can suppose that A is upper-triangular [or diagonal].

Suppose $e_1, \dots, e_n$ is an eigenbasis of A :

$$A e_i = \lambda_i e_i.$$

$S^k V$ is spanned by $\overbrace{e_{i_1} \dots e_{i_k}}^{\text{again an eigenbasis!}}$.

$$A(e_{i_1} \dots e_{i_k}) = (\lambda_{i_1} \dots \lambda_{i_k}) e_{i_1} \dots e_{i_k}.$$

Same about $\Lambda^k V$.

k -th complete symmetric polynomial

$$\text{so } \text{tr } S^k V = \sum_{1 \leq i_1 \leq \dots \leq i_k \leq n} \lambda_{i_1} \dots \lambda_{i_k} = h_k(\lambda_1, \dots, \lambda_n)$$

↓
elementary sym. polynomial

$$\text{tr } \Lambda^k V = \sum_{1 \leq i_1 < \dots < i_k \leq n} \lambda_{i_1} \dots \lambda_{i_k} = e_k(\lambda_1, \dots, \lambda_n).$$

$$\text{tr } A = \lambda_1 + \dots + \lambda_n$$

$$\text{tr } A^2 = \lambda_1^2 + \dots + \lambda_n^2$$

$$\text{tr } A^k = \lambda_1^k + \dots + \lambda_n^k = p_k(\lambda_1, \dots, \lambda_n)$$

Newton power sums.

Question: express h_k and e_k via $p_1, \dots, p_k$.

Characters of representations

G - finite group

V is a finite-dim complex representation

A corollary from the 1st proof of Maschke's thm.

$\rho: G \rightarrow GL(V)$ homomorphism.

There exists a G -invariant scalar product on V :

$(\cdot, \cdot)$ s.t. $\forall g \in G, v, w \in V$

$$(\rho(g)v, \rho(g)w) = (v, w), \quad (\cdot, \cdot) \text{ is}$$

positive def. $\frac{1}{2}$ linear form:

$$(v, w) = \overline{(w, v)}, \quad (\lambda \cdot v, w) = \overline{\lambda} (v, w) \\ (v, \lambda \cdot w) = \lambda \cdot (v, w) \quad \lambda \in \mathbb{C}.$$

$\rho(g)$ given by a unitary matrix.

↳ • diagonalizable

• all the eigenvalues of $\rho(g)$ are roots of 1:
 $|\lambda_i| = 1$.

How to define a diag. linear operator A ?

multiset $\{\lambda_1, \dots, \lambda_n\}$.

$$\lambda_1 + \dots + \lambda_n = \text{tr } A;$$

$$\lambda_1^2 + \dots + \lambda_n^2 = \text{tr } A^2;$$

$$\lambda_1^n + \dots + \lambda_n^n = \text{tr } A^n.$$

G is a group, so knowing $\text{tr } \rho(g) \quad \forall g \in G$,
we can recover the eigenvalues of each $\rho(g)$.

(we know $\text{tr } \rho(g), \text{tr } \rho(g^2), \dots$)

Def. Let $\rho: G \rightarrow GL(V)$ be a repr.

The character χ_ρ or χ_V is a function $G \rightarrow \mathbb{C}$

$$\chi_\rho(g) = \text{tr } \rho(g).$$

Example: Let $\rho: G \rightarrow \mathbb{C}^\times$ be the trivial rep.
 $\chi_\rho \equiv 1.$

A not-so-stupid example:

$G \curvearrowright X$ finite set

$$V_X = \langle e_x \mid x \in X \rangle$$

$$\rho(g) e_x = e_{g \cdot x}.$$

$$\chi_V(g) = \# \{ x \in X \mid g \cdot x = x \} = \# X^g.$$

$$\rho(g) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

perm.
matrix

Rem. $\chi_V(e) = \dim V = \text{tr } \text{Id}_V.$

A really important example: $G \curvearrowright G$ left shifts

$$R: G \longrightarrow GL(n, \mathbb{C}), \quad n = |G|.$$

perm. rep. for this action: **Regular rep.**

$$\chi_R(e) = |G| = n.$$

$$\chi_R(g) = 0 \quad g \neq e$$

Left shift has no fixed pts: $g \cdot h = h \Rightarrow g = e$
at $g \neq e$

Prop. Let V, W be two reps of G . Then:

$$(a) \quad \chi_{V \oplus W}(g) = \chi_V(g) + \chi_W(g)$$

$$(b) \quad \chi_{V \otimes W}(g) = \chi_V(g) \cdot \chi_W(g)$$

$$(c) \quad \chi_{V^*}(g) = \overline{\chi_V(g)}.$$

Proof: (a): $A: V \rightarrow V$ | $A \oplus B: V \oplus W \rightarrow V \oplus W$
 (b) $B: W \rightarrow W$ | eigenvalues:
 $\{\lambda_1, \dots, \lambda_n, \mu_1, \dots, \mu_m\}$
 (as a multiset)

$$\text{tr } A \oplus B = \text{tr } A + \text{tr } B.$$

$$\text{tr } A \otimes B = \text{tr } A \cdot \text{tr } B.$$

(c) $A: V \rightarrow V$ has eigenvalues $\lambda_1, \dots, \lambda_n$

$A^*: V^* \rightarrow V^*$ has $\lambda_1^{-1}, \dots, \lambda_n^{-1}$.

$$\begin{aligned} \lambda_i^{-1} &= \overline{\lambda_i}, & \text{tr } A^* &= \lambda_1^{-1} + \dots + \lambda_n^{-1} = \overline{\lambda_1} + \dots + \overline{\lambda_n} = \\ & & &= \overline{(\lambda_1 + \dots + \lambda_n)} = \overline{\text{tr } A} \end{aligned}$$

$$\rho: G \longrightarrow GL(V)$$

$$V^* = \text{Hom}(V, \mathbb{C})$$

Dual representation:

How to define $\rho^*: G \longrightarrow GL(V^*)$?

$$\varphi \in V^*$$

$$v \in V$$

$$\langle \varphi, v \rangle = \varphi(v).$$

Want $\langle, \rangle$ to be invariant under the action of G .

$$\langle \rho^*(g) \varphi, \rho(g) v \rangle = \varphi(v).$$

$e_1, \dots, e_n$ basis in V

$\xi_1, \dots, \xi_n$ basis in V^*

$\rho^*(g)$ is given by $\rho(g^{-1})^t$.

$$\langle \xi_i, e_j \rangle = \delta_{ij}$$

$$\langle \rho^*(g) \xi_i, \rho(g) e_j \rangle = \delta_{ij}$$

$$(\rho^*(g))^T \cdot \rho(g) = I. \quad \rho(g) = (\rho(g)^{-1})^T$$

$$\text{If } \rho(g) = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}, \text{ then } \rho^*(g) = \begin{pmatrix} \lambda_1^{-1} & & 0 \\ & \ddots & \\ 0 & & \lambda_n^{-1} \end{pmatrix} =$$

$$= \begin{pmatrix} \overline{\lambda_1} & & \\ & \ddots & \\ & & \overline{\lambda_n} \end{pmatrix}.$$