

Tensor products of vector spaces

Lecture 3

12/09/22

Multilinear maps

K ground field

$V_1, \dots, V_k, W$

Def. A map $\varphi: V_1 \times \dots \times V_k \rightarrow W$ is multilinear if

$$\varphi(\dots, \lambda v_i + \mu v_i', \dots) = \lambda \varphi(\dots, v_i, \dots) + \mu \varphi(\dots, v_i', \dots)$$

$k=1$: linear maps

$k=2, W=K$: bilinear forms.

Multilinear maps form a vector space

$\text{Hom}(V_1, \dots, V_k; W)$.

Prop. If $\dim V_i = d_i$, $\dim W = d$, then

$$\dim \text{Hom}(V_1, \dots, V_k; W) = d_1 \dots d_k \cdot d.$$

Proof: Let $e_1^{(i)}, \dots, e_{d_i}^{(i)}$ be a basis in V_i
 $e_1, \dots, e_d$ ——— in W .

To define φ , we need to define it only on
 $\varphi(e_{\alpha_1}^{(1)}, e_{\alpha_2}^{(2)}, \dots, e_{\alpha_k}^{(k)})$, $\alpha_i \in [1, d_i]$.

Indeed: if $v_i = \sum x_{\alpha_i}^{(i)} e_{\alpha_i}^{(i)}$, then

$$\varphi(v_1, \dots, v_k) = \sum_{\substack{1 \leq \alpha_1 \leq d_1 \\ \vdots \\ 1 \leq \alpha_k \leq d_k}} x_{\alpha_1}^{(1)} \dots x_{\alpha_k}^{(k)} \varphi(e_{\alpha_1}^{(1)}, \dots, e_{\alpha_k}^{(k)}).$$

Expand each $\varphi(e_{\alpha_1}^{(1)}, \dots, e_{\alpha_k}^{(k)})$ in the basis
 of W :
 get a "matrix" of dim $k+1$
 of size $d_1 \times d_2 \times \dots \times d_k \times d$. □

A basis of $\text{Hom}(V_1, \dots, V_k; W)$ is formed by

$$\delta_{(i_1, \dots, i_k)}^j : (e_{\alpha_1}^{(1)} \dots e_{\alpha_k}^{(k)}) \mapsto \begin{cases} e_j, & \text{if } (i_1, \dots, i_k) \\ & (\alpha_1, \dots, \alpha_k) \\ 0 & \text{otherwise.} \end{cases}$$

$$1 \leq j \leq d, \quad 1 \leq i_r \leq d_r.$$

Tensor products.

$$V = \langle e_i \mid i \in I \rangle$$

$$W = \langle f_j \mid j \in J \rangle$$

U another vector space

I, J need not be finite --

Prop. Let $\varphi: V \times W \rightarrow U$ be a bilinear map.

TFAE: (1) $\varphi(e_i, f_j)$ form a basis of U ;

(2) $\forall u \in U$ there is a unique presentation

$$u = \sum_{i \in I} \varphi(e_i, w_i), \text{ where } w_i \in W.$$

(3) $\forall u \in U$ there is a unique presentation

$$u = \sum_{j \in J} \varphi(v_j, f_j).$$

Proof. (1) $\Leftrightarrow$ (2). Indeed, if $u = \sum_{\substack{i \in I \\ j \in J}} x_{ij} \varphi(e_i, f_j).$

Then $u = \sum \varphi(e_i, w_i)$, where $w_i = \sum_{j \in J} x_{ij} f_j.$

(use linearity of φ w.r.t. 2nd argument!)
(1) $\Leftrightarrow$ (3) similarly.

Corollary. Condition (1) does not depend upon a choice of bases in V, W .

Proof (1) $\Leftrightarrow$ (3) indep. of a basis in V ; same about (1) $\Leftrightarrow$ (2).

Def. A vector space T together with a bilinear map

$$\otimes: V \times W \longrightarrow T$$

satisfying the condition: for $\{e_i \mid i \in I\}$ and $\{f_j \mid j \in J\}$ bases of V and W resp., $e_i \otimes f_j$ is a basis in T , is called (a) tensor product of V and W .

Indep. upon a choice of bases in V and W :
previous corollary.

Existence: $T = \langle t_{ij} \mid i \in I, j \in J \rangle$, $e_i \otimes f_j \mapsto t_{ij}$.

Prop: The space T and the map $\otimes$ is unique in the following sense: if $(T_1, \otimes_1)$ and $(T_2, \otimes_2)$ are two such tensor products of V and W , then $\exists!$ isomorphism $\psi: T_1 \rightarrow T_2$,

$$\psi(v \otimes_1 w) = v \otimes_2 w$$

Proof: define ψ on basis vectors by $\psi(e_i \otimes_1 f_j) = e_i \otimes_2 f_j$. $\square$

Notation: denote T by $V \otimes W$.

If V, W are finite dim'l, $\dim V \otimes W = \dim V \cdot \dim W$.

Example: $V = \mathbb{K}[x]$, $W = \mathbb{K}[y]$. $\mathbb{K}[x] \otimes \mathbb{K}[y] = \mathbb{K}[x, y]$
 $(x^n, y^m) \mapsto x^n y^m$. $f(x) \otimes g(y) \mapsto f(x) \cdot g(y)$.

$$\begin{array}{ccc}
 V \times W & \xrightarrow{\varphi} & U \\
 \searrow \otimes & & \nearrow \exists! F \\
 & V \otimes W &
 \end{array}$$

Universal property of tensor product:

Prop: For each bilinear map $V \times W \xrightarrow{\varphi} U$,
 $\exists!$ linear map $F: V \otimes W \rightarrow U$,
 s.t. $F(v \otimes w) = \varphi(v, w)$.

Proof: define $F(e_i \otimes f_j) := \varphi(e_i, f_j)$.

□

More examples of tensor products.

From now on: V, W finite dim'l:

$$V = \langle e_1, \dots, e_m \rangle \quad V^* = \langle \xi_1, \dots, \xi_m \rangle \quad \xi_i(e_j) = \delta_{ij}$$

$$W = \langle f_1, \dots, f_n \rangle \quad W = \langle \eta_1, \dots, \eta_n \rangle \quad \eta_i(f_j) = \delta_{ij}$$

$$V^* \times W \longrightarrow \text{Hom}(V; W)$$

$$(\overset{V}{\alpha}, \overset{W}{w}) \longmapsto \overset{W}{\alpha \otimes w} : \quad (\alpha \otimes w)(v) = \alpha(v) \cdot w$$

$$\text{Ker}(\alpha \otimes w) = \text{Ker } \alpha \quad \text{Im}(\alpha \otimes w) = \langle w \rangle.$$

rank 1 linear map; every rank 1 map has the form $\alpha \otimes w$.

Corresponds to certain linear map

$$V^* \otimes W \longrightarrow \text{Hom}(V, W).$$

Claim: This is an isomorphism!

Consider $\xi_i \otimes f_j$. It corr. to a matrix

E_{ji} . (1 on the (j, i) -th place, 0's
on other places).

$\uparrow$
form a basis of $\text{Hom}(V, W)$.

Isomorphism!

$$V^* \otimes W \simeq \text{Hom}(V, W)$$

Exercise.* Using the universality property as a definition, show that $V \otimes W$ is well-defined (exists and is unique).

↳ "Real life" example of $\otimes$: field extensions.

$\mathbb{K} \subset \mathbb{L}$ (think about $\mathbb{R} \subset \mathbb{C}$!)
subfield field

$\mathbb{L}$ is a vector space over $\mathbb{K}$!

V vector space over $\mathbb{K}$.

$V(\mathbb{L}) := \mathbb{L} \otimes_{\mathbb{K}} V$ $\mathbb{K}$ -vector space

Define the mult. by $\lambda \in \mathbb{L}$ as follows:

$$\lambda \cdot (\mu \otimes v) = (\lambda \cdot \mu) \otimes v$$

$\nwarrow$ product in $\mathbb{L}$!

$V \hookrightarrow V(\mathbb{L})$ as v.sp. over $\mathbb{K}$:

$$v \mapsto 1 \otimes v$$

If $e_1, \dots, e_n$ is a $\mathbb{K}$ -basis of V , then

$1 \otimes e_1, \dots, 1 \otimes e_n$ is an $\mathbb{L}$ -basis of $V(\mathbb{L})$

$$\dim_{\mathbb{K}} V = \dim_{\mathbb{L}} V(\mathbb{L}).$$

$V(\mathbb{L})$ can be viewed as a $\mathbb{K}$ -vector space!

$$\dim_{\mathbb{K}} V(\mathbb{L}) = \dim_{\mathbb{K}} V \cdot \dim_{\mathbb{K}} \mathbb{L}.$$

$\nwarrow$ complexification of V

In particular: $\mathbb{L} = \mathbb{C}$, $\mathbb{K} = \mathbb{R}$: $V(\mathbb{C}) \ni v + iw$, $v, w \in V$.

Tensor products of linear operators.
 V, W

$$A: V \rightarrow V, \quad B: W \rightarrow W.$$

$$A \otimes B: V \otimes W \rightarrow V \otimes W.$$

$$(A \otimes B)(v \otimes w) = Av \otimes Bw.$$

Let A be given by A in the basis $e_1, \dots, e_m$ of V
 $\leftarrow B \quad \leftarrow B \quad \leftarrow B \quad \leftarrow f_1, \dots, f_n$ of W .

$$e_1 \otimes f_1, e_1 \otimes f_2, \dots, e_1 \otimes f_n, e_2 \otimes f_1, \dots, e_2 \otimes f_n, \dots, e_m \otimes f_1, \dots, e_m \otimes f_n$$

basis of $V \otimes W$.

$A \otimes B$ given by the matrix

$$\begin{pmatrix}
 a_{11}B & a_{12}B & \dots & a_{1n}B \\
 a_{21}B & a_{22}B & \dots & a_{2n}B \\
 \vdots & \vdots & \ddots & \vdots \\
 a_{m1}B & \dots & \dots & a_{mn}B
 \end{pmatrix}$$

$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix}.$

$$\text{tr}(A \otimes B) = \text{tr} A \cdot \text{tr} B.$$

Exercise. Express $\det(A \otimes B)$, knowing $\det A$
 and $\det B$.

Properties of tensor products.

Prop. There exist canonical isomorphisms
(not involving bases)

$$(1) \quad V \otimes W \simeq W \otimes V \quad \text{commutativity}$$

$$(2) \quad V \otimes (W \otimes U) \simeq (V \otimes W) \otimes U \quad \text{associativity}$$

$$(3) \quad V \otimes (W \oplus U) \simeq (V \otimes W) \oplus (V \otimes U) \quad \text{distributivity}$$

Proof. Left as exercise.

We can define $V_1 \otimes V_2 \otimes \dots \otimes V_k$; it does not depend upon the order of mult.

For this product, there are analogues of properties shown for $\otimes$ of two spaces.

In part, $\text{Hom}(V_1 \otimes \dots \otimes V_k; W) \simeq \text{Hom}(V_1, \dots, V_k; W)$

$$F(v_1 \otimes \dots \otimes v_k) := \varphi(v_1, \dots, v_k)$$

is an isomorphism.

Def. Elements of the form $v_1 \otimes \dots \otimes v_k$, $v_i \in V_i$,
 $\uparrow$
 $V_1 \otimes \dots \otimes V_k$
are called decomposable tensors.

Rem. The sum of two decomp. tensors is not necessarily decomposable!

Example: $V \overset{*}{\otimes} W \simeq \text{Hom}(V; W)$

$\alpha \otimes w$ - linear maps of rank 1.

Notation: $V^{\otimes k} := \underbrace{V \otimes \dots \otimes V}_{k \text{ copies}}$

Symmetric powers of vector spaces.

Def. V, U vector spaces
 A k -linear map $V \times \dots \times V \longrightarrow U$ is symmetric
 if $\varphi(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = \varphi(v_1, \dots, v_k)$.

Def A vector space S with a symmetric k -linear map $\underbrace{V \times \dots \times V}_k \rightarrow S$ ω
 $(v_1, \dots, v_k) \mapsto v_1 \cdot \dots \cdot v_k$

is said to be the k -th symmetric power of V ,
 if for some basis $e_1, \dots, e_n$ of V ,
 the elements $e_{i_1} \dots e_{i_k}$, $1 \leq i_1 \leq i_2 \leq \dots \leq i_k \leq n$,
 form a basis of S .

Prop. Such an S exists and is independent
 upon a choice of basis in V .

Proof. Existence: construction.

$$S = \langle f_{i_1 i_2 \dots i_k} \mid 1 \leq i_1 \leq \dots \leq i_k \leq n \rangle.$$

$$V \times \dots \times V \longrightarrow S$$

$$(e_{i_1} \dots e_{i_k}) \mapsto f_{i_1 \dots i_k}$$

Exercise. $\dim S^k V = \binom{n+k-1}{k}$

Exercise. Formulate and prove uniqueness for S .

Notation: $S^k V := S$. k -th symmetric power of V .

Let $A: V \rightarrow V$.

$S^k A: S^k V \rightarrow S^k V$.

$v_1 \cdots v_k \mapsto A v_1 \cdots A v_k$.

Exercise. (1) Show that $\text{tr } S^2 A = \frac{1}{2}((\text{tr } A)^2 + \text{tr } A^2)$.

(2) Find a formula for $\text{tr } S^3 A$.