

Lecture 2

08/09/2022

G - group

V finite dim'l vector space

over $\mathbb{K}$ (mostly we'll have $\mathbb{K} = \mathbb{C}$).

Def: Repr. of G in the space V is

homomorphism $\rho: G \rightarrow GL(V)$

$$\rho: g \mapsto \rho(g)$$

Category: • Objects

• Morphisms: "nice" maps between objects.

What are morphisms between reps. of a group G ?

Let V, W be reps of G ;

$$\begin{array}{ccc} G & \xrightarrow{\rho_1} & GL(V) \\ & \searrow \rho_2 & \\ & & GL(W) \end{array} \quad \begin{array}{l} \text{let } \varphi: V \rightarrow W \\ \text{be a linear map.} \end{array}$$

Def. φ is said to be a G -morphism, or

- morphism of reps $\leftarrow$ 'same thing.'
- intertwining operator

if

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & W \\ \rho_1(g) \downarrow & & \downarrow \rho_2(g) \\ V & \xrightarrow{\varphi} & W \end{array} \quad \begin{array}{l} \text{is commutative, i.e.} \\ \rho_2(g) \circ \varphi = \varphi \circ \rho_1(g) \quad \forall g \in G. \end{array}$$

Define $\text{Ker } \varphi \subset V$

kernel $\{v \in V \mid \varphi(v) = 0\}$

$\text{Im } \varphi \subset W$ image.

$\{w \in W \mid \exists v \in V: \varphi(v) = w\}$

Prop. For φ morphism, $\text{Ker } \varphi \subset V$ and $\text{Im } \varphi \subset W$ are subrepresentations.

Pf: Let $v \in \text{Ker } \varphi$. $\varphi(v) = 0$.

Need to show: $\forall g \in G, \rho_1(g) \cdot v \in \text{Ker } \varphi$

$$\varphi(\rho_1(g) \cdot v) \stackrel{?}{=} 0$$

Exercise. Show that

$\text{Im } \varphi$ is a subrep in W . $\rho_2(g) \circ \varphi(v) = 0$.

Thm. Let G be a finite group,

$\rho: G \rightarrow GL(V)$ fin. dim'l rep, $\text{char } K \nmid |G|$
(or $\text{char } K = 0$)

Then V is completely reducible: $\forall$ subrep $U \subset V$,
 $\exists$ subrep $W \subset V$,
s.t. $V = U \oplus W$.

Pf: $\pi_0: V \rightarrow V$ is said to be a projector, if $\pi_0^2 = \pi_0$.

$$\pi_0|_{\text{Im } \pi_0} = \text{Id}. \quad \text{Ker } \pi_0 \oplus \text{Im } \pi_0 = V$$

Let π_0 be an arb. projector $V \rightarrow U \subset V$.
($\text{Im } \pi_0 = U$).

Generally speaking, $\text{Ker } \pi_0$ is not necess. a subrep.

Consider
$$\pi(v) = \frac{1}{|G|} \sum_{g \in G} \underset{\rho(g)}{\underset{\parallel}{g}} \left(\underset{\rho(g)^{-1}}{\overset{\pi}{\pi_0}}(g^{-1}(v)) \right).$$

$\pi: V \rightarrow V$

Claim: π is a morphism of reps.

Indeed: need to show: $\rho(h) \circ \pi = \pi \circ \rho(h).$

$$\rho(h) \circ \pi \circ \rho(h)^{-1} = \pi.$$

$$\begin{aligned} h \pi h^{-1}(v) &= \frac{1}{|G|} \sum_{g \in G} h g \pi_0 g^{-1} h^{-1} v = \frac{1}{|G|} \sum_{g \in G} (hg) \pi_0 (hg)^{-1} v = \\ &= \frac{1}{|G|} \sum_{\bar{g} \in G} \bar{g} \pi_0 \bar{g}^{-1} v = \pi. \end{aligned}$$

This means that $\text{Ker } \pi$ is subrep!

π is a projector onto U .

$$\text{So } V = \text{Ker } \pi \oplus U$$

$\underbrace{\hspace{1.5cm}}_W$

Remark about semisimplicity: Abelian groups:

$$\mathbb{Z}/4\mathbb{Z} \supset \mathbb{Z}/2\mathbb{Z},$$

$$\frac{\mathbb{Z}/4\mathbb{Z}}{\mathbb{Z}/2\mathbb{Z}} \cong \mathbb{Z}/2\mathbb{Z}.$$

But $\mathbb{Z}/4\mathbb{Z} \not\cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}!$ Abelian groups are not nice.

Reps of finite groups are nice!

(The category Rep_G is semisimple:

every rep. is a direct sum of irreducible representations.)

irreducible representation $\rightarrow$ irrep.

Schur's Lemma. Let V, W be two ^{irreducible} reps of G .

$\varphi: V \rightarrow W$. Then (a) φ is isom. or $\varphi = 0$;
morphism of G -reps

(b) If $K = \mathbb{C}$, $V = W$,
then $\varphi: V \rightarrow V$ $\varphi = \lambda \cdot \text{Id}$.

Proof: (a)
$$\begin{array}{ccc} V & \xrightarrow{\varphi} & W \\ \downarrow & & \downarrow \\ \text{Ker } \varphi & & \text{Im } \varphi \end{array}$$

↑ subspaces!

• $\text{Ker } \varphi$ is either V
 $(\varphi \equiv 0)$
 or $\text{Ker } \varphi = 0$
 $(\varphi \text{ is a monomorphism})$

• $\text{Im } \varphi = 0$ $(\varphi \equiv 0)$
 or $\text{Im } \varphi = W$ $(\varphi \text{ is an epimorphism})$

+
 φ is an epimorphism

 φ is an isomorphism.

(b) $\varphi: V \rightarrow V$
 $\lambda \text{Id}: V \rightarrow V$
 $\lambda \in \mathbb{C}$

$\varphi - \lambda \text{Id}: V \rightarrow V$ is a morphism for
 any $\lambda \in \mathbb{C}$.

φ has an eigenvector: $v \in V$
 $\varphi v = \lambda v$.

$(\varphi - \lambda \text{Id})v = 0$ $\text{Ker}(\varphi - \lambda \text{Id}) \neq 0$. So $\text{Ker}(\varphi - \lambda \text{Id}) = V$. $\varphi = \lambda \text{Id}$.

Back to pf. of Maschke's thm:

$$\pi_0: V \longrightarrow U \subset V \quad \text{projector}$$

$$\pi = \frac{1}{|G|} \sum_{g \in G} g \pi_0 g^{-1}$$

Know that $\text{Im } \pi_0 = U$

$$\pi_0|_U = \text{Id.}$$

Need to check: ~~$\pi^2 = \pi$~~

$$\bullet \text{Im } \pi = \text{Im } \pi_0 = U$$

$$\bullet \pi|_U = \text{Id.}$$

$u \in U$ Apply π :

$$\pi u = \frac{1}{|G|} \cdot \sum_{g \in G} g \pi(g^{-1}u) =$$

$$= \frac{1}{|G|} \sum_{g \in G} g \cdot g^{-1}u = \frac{1}{|G|} \sum_{g \in G} u = u.$$

u , since U is
subrep.

Show that $\text{Im } \pi \subset U$.

$$\pi v = \frac{1}{|G|} \sum_{g \in G} g \cdot (\pi \circ g^{-1} v) \in U.$$

Prop. For each rep. V of a finite grp G ,
there is a decomp.

$$V = V_1^{\oplus a_1} \oplus V_2^{\oplus a_2} \oplus \dots \oplus V_k^{\oplus a_k} \quad (*)$$

The summands V_i and their mult. a_i are
uniquely determined.

Pf. Existence : from Maschke's theorem.

Uniqueness: $W = \bigoplus W_j^{\oplus b_j}$

$$\varphi: \bigoplus V_i^{\oplus a_i} \longrightarrow \bigoplus W_j^{\oplus b_j},$$

Each summand $V_i^{\oplus a_i}$ is mapped into $W_j^{\oplus b_j}$ for

$W_j \simeq V_i$. (Schur's lemma).

Apply this to the identity map $\text{Id}: V \rightarrow V$.

□

Sometimes we'll write $V = a_1 V_1 \oplus \dots \oplus a_k V_k$.

Three fundamental problems of rep. theory:

(1) Given a group G , classify all its irreps

(2) Given a rep V , decompose it into irreps:

$$V = \bigoplus V_i^{\oplus a_i}.$$

(3) "Plethysm". For V, W irreps,
solve (2) for $V \otimes W = \bigoplus \text{irreps}$

$$S^k V, \wedge^k V, \dots$$

Representations of Abelian (=commutative) groups.

$G \curvearrowright V \quad \forall g \in G: V \xrightarrow{\rho(g)} V.$ not necessarily a G -morphism!

Is a morphism if $\rho(h) \rho(g) \rho(h)^{-1} = \rho(g) \quad \forall h \in G.$

$\Leftrightarrow$ if $\rho(h) \rho(g) = \rho(g) \rho(h)$; true if $hg = gh \quad \forall h \in G$
i.e. if $g \in Z(G).$

If G Abelian, V irred: $\rho(g): V \rightarrow V$ morphism.

Schur's Lemma: $\rho(g) = \lambda \cdot \text{Id.} \quad \forall g \in G.$

Any subspace $U \subset V$ is G -invar, V irred. $\Rightarrow$

$$\Rightarrow \dim V = 1.$$

Thm. Each irrep of G abelian has $\dim. 1$.

$$\rho: G \longrightarrow \mathbb{C}^* \quad \text{characters of } G.$$

They form a group If $|G| < \infty$,

$$G^V \cong G \quad \text{no canonical isom.} \quad (\rho_1 \rho_2)(g) = \rho_1(g) \cdot \rho_2(g)$$

S_3 : permutations of $\{1, 2, 3\}$.

What irreps of S_3 are there?

1-dim'l: $U: S_3 \longrightarrow \mathbb{C}^*, \quad w \mapsto 1. \quad \text{Trivial.}$

$U': S_3 \longrightarrow \mathbb{C}^* \quad w \mapsto \text{sgn } w \in \{\pm 1\}$

The sign rep.

2-dim'l: V' tautological rep.

$$S_3 \curvearrowright \mathbb{C}^3 = V \quad \text{by coord. perm.}$$

$$\langle \underset{U}{(1, 1, 1)} \rangle \simeq U \quad \text{trivial}$$

$$V \simeq U \oplus V'$$

$$\text{Its } \underline{\text{complement}} \text{ is } V' = \{ (x_1, x_2, x_3) \mid x_1 + x_2 + x_3 = 0 \} \subset V$$

Exercise. Prove V' is an irrep.

$$S_3 \supset A_3 = \langle e, \tau = (123), \tau^2 \rangle \quad \text{Abelian subgroup.}$$

$$\tau^3 = \text{Id.}$$

$$S_3 \curvearrowright W = \langle w_i \mid \tau w_i = \lambda_i w_i \rangle. \quad \lambda^3 = 1.$$

$$\lambda \in \{1, \omega, \omega^2\}.$$

$$\sigma = (12) \quad \sigma \tau \sigma^{-1} = \sigma \tau \sigma = \tau^2.$$

$$\sigma \tau = \tau^2 \sigma.$$

$$\omega = -\frac{1}{2} + \frac{\sqrt{3}i}{2} = \sqrt[3]{1}.$$

w_i eigenvector for τ .

$$\sigma \tau^2 = \tau \sigma.$$

Claim: σw_i is also τ -eigenvector! with eigenvalue λ_i^2 .

$$\tau \sigma w_i = \sigma \tau^2 w_i = \sigma \cdot \lambda_i^2 w_i = \lambda_i^2 \sigma w_i$$

Case 1: If $\tau w_i = \lambda_i w_i$, $\lambda_i \neq 1$.

$$\tau(\sigma w_i) = \lambda_i^2 (\sigma w_i) \quad \lambda_i \neq \lambda_i^2.$$

$\langle w_i, \sigma w_i \rangle$ span a 2-dim'l repr.

Exercise. Show it is isom. to V' .

Case 2: if $\lambda_i = 1$, $\tau w_i = w_i$

σw_i is either $\sim w_i$ or lin. indep.

2a:
$$\left. \begin{array}{l} \sigma w_i = \pm w_i, \\ \tau w_i = w_i \end{array} \right\} \langle w_i \rangle \text{ spans a 1 dim'l ssp } U \text{ or } U'.$$

2b

$\sigma w_i \neq w_i$
 τ acts triv.

$$w_i \xrightleftharpoons[\sigma]{\sigma} \sigma w_i$$

$$\langle w_i, \sigma w_i \rangle$$

$\cup$

$\supset$

$\tilde{w}$

$$\langle w_i + \sigma w_i \rangle$$

$$\langle w_i - \sigma w_i \rangle$$

$$\sigma \tilde{w} = -\tilde{w}$$

$\parallel$

U

$\parallel$

U'

Exercise:

decompose the reg. rep. of S_3 into irreps.

U, U', V' 2dim'l
 1 dim'l.