

Representation Theory

Evgeny Smirnov

esmirnov@hse.ru

Mon 15:10-17:00

Thu 18:40-20:30

Sino-Russian Mathematics Center

Peking University

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What does Representation Theory study?

Lecture 1
05/09/22

- Groups act on -sets

- vector spaces

- manifolds

- Linear algebra:

How do groups act on vector spaces?

finite groups

discrete groups

Lie groups
 $GL(n)$, $O(n)$

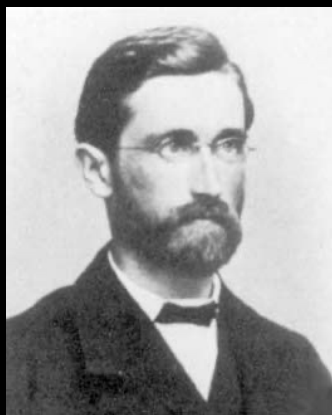
over various fields: $\mathbb{C}$, $\mathbb{R}$, $\mathbb{F}_q, \dots$

• fin.-dim

• inf.-dim.

• add. structure

Starting point for Representation Theory.



Richard Dedekind



Georg Frobenius

square table $|G| \times |G|$

• replace g_i by x_i .

• get matrix C .

• $\Delta = \det C$.

Let G be a finite group.

Take the Cayley table of the group G .
(the mult. table of G).

	g_2
g_1	$g_1 g_2$

$$\Delta = \prod f_i^{d_i}, \quad \deg f_i = d_i. \quad ???$$

Example: $G = S_3$

e	(123)	(132)	(12)	(23)	(13)
(123)	(132)	e	(13)	(12)	(23)
(132)	e	(123)	(23)	(13)	(12)
(12)	(23)	(13)	e	(123)	(132)
(23)	(13)	(12)	(132)	e	(123)
(13)	(12)	(23)	(123)	(132)	e

e	t	s	x	y	z
t	s	e	z	x	y
s	e	t	y	z	x
x	y	z	e	t	s
y	z	x	s	e	t
z	x	y	t	s	e

Compute the determinant!

(det[[e,t,s,x,y,z],[t,s,e,z,x,y],[s,e,t,y,z,x],[x,y,z,e,t,s],[y,z,x,s,e,t],[z,x,y,t,s,e]])

 NATURAL LANGUAGE

 MATH INPUT

 EXTENDED KEYBOARD

 EXAMPLES

 UPLOAD

 RANDOM

Input interpretation

$$\begin{vmatrix} e & t & s & x & y & z \\ t & s & e & z & x & y \\ s & e & t & y & z & x \\ x & y & z & e & t & s \\ y & z & x & s & e & t \\ z & x & y & t & s & e \end{vmatrix}$$

|m| is the determinant

Result

$$\begin{aligned} & e(t s^4 - x y s^3 - x z s^3 - y z s^3 - e^2 s^3 - 3 e t^2 s^2 - t x^2 s^2 + e x^2 s^2 - t y^2 s^2 + \\ & e y^2 s^2 - t z^2 s^2 + e z^2 s^2 + t x y s^2 + 2 e x y s^2 + t x z s^2 + 2 e x z s^2 + \\ & t y z s^2 + 2 e y z s^2 + t^4 s + x y^3 s + x z^3 s + y z^3 s - t^2 x^2 s - \\ & 2 e t x^2 s - t^2 y^2 s - x^2 y^2 s - 2 e t y^2 s - t^2 z^2 s - x^2 z^2 s - y^2 z^2 s - \\ & 2 e t z^2 s - x y z^2 s + 4 e^3 t s + x^3 y s + t^2 x y s + 2 e t x y s - \\ & 3 e^2 x y s + x^3 z s + y^3 z s - x y^2 z s + t^2 x z s + 2 e t x z s - \\ & 3 e^2 x z s + t^2 y z s - x^2 y z s + 2 e t y z s - 3 e^2 y z s - e x^4 - e y^4 - \\ & e z^4 - e^2 t^3 + t x y^3 + t x z^3 + t y z^3 + e t^2 x^2 + 2 e^3 x^2 + e t^2 y^2 - \\ & t x^2 y^2 - e x^2 y^2 + 2 e^3 y^2 + e t^2 z^2 - t x^2 z^2 - e x^2 z^2 - t y^2 z^2 - \\ & e y^2 z^2 - t x y z^2 + 2 e x y z^2 + 2 e^3 z^2 + t x^3 y - t^3 x y + 2 e t^2 x y - \\ & 3 e^2 t x y + t x^3 z + t y^3 z - t x y^2 z + 2 e x y^2 z - t^3 x z + 2 e t^2 x z - \\ & 3 e^2 t x z - t^3 y z + 2 e t^2 y z - t x^2 y z + 2 e x^2 y z - 3 e^2 t y z - e^5) - \\ & s(s^5 - 2 x^2 s^3 - 2 y^2 s^3 - 2 z^2 s^3 - 4 e t s^3 + t^3 s^2 + 3 t x y s^2 + 3 e x y s^2 + \\ & 3 t x z s^2 + 3 e x z s^2 + 3 t y z s^2 + 3 e y z s^2 + e^3 s^2 + x^4 s + y^4 s + \\ & z^4 s + 3 e^2 t^2 s - t^2 x^2 s + 2 e t x^2 s - e^2 x^2 s - t^2 y^2 s + x^2 y^2 s + \\ & 2 e t y^2 s - e^2 y^2 s - t^2 z^2 s + x^2 z^2 s + y^2 z^2 s + 2 e t z^2 s - 2 x y z^2 s - \\ & e^2 z^2 s - 2 t^2 x y s - 2 e t x y s - 2 e^2 x y s - 2 x y^2 z s - 2 t^2 x z s - \\ & 2 e t x z s - 2 e^2 x z s - 2 t^2 y z s - 2 x^2 y z s - 2 e t y z s - 2 e^2 y z s - \\ & e t^4 - t x y^3 - e x y^3 - t x z^3 - e x z^3 - t y z^3 - e y z^3 + e t^2 x^2 + \\ & e^2 t x^2 + e t^2 y^2 + t x^2 y^2 + e x^2 y^2 + e^2 t y^2 + e t^2 z^2 + t x^2 z^2 + \\ & e x^2 z^2 + t y^2 z^2 + e y^2 z^2 + e^2 t z^2 + t x y z^2 + e x y z^2 - e^4 t - \\ & t x^3 y - e x^3 y + t^3 x y - e t^2 x y - e^2 t x y + e^3 x y - t x^3 z - e x^3 z - \\ & t y^3 z - e y^3 z + t x y^2 z + e x y^2 z + t^3 x z - e t^2 x z - e^2 t x z + \\ & e^3 x z + t^3 y z - e t^2 y z + t x^2 y z + e x^2 y z - e^2 t y z + e^3 y z) + \\ & t(-t^5 + 2 x^2 t^3 + 2 y^2 t^3 + 2 z^2 t^3 + 4 e s t^3 - s^3 t^2 - 3 s x y t^2 - 3 e x y t^2 - \end{aligned}$$

Expanded form

$$\begin{aligned} & -s^6 + 3x^2s^4 + 3y^2s^4 + 3z^2s^4 + 6ets^4 - 2t^3s^3 - 6txys^3 - 6exys^3 - \\ & 6txzs^3 - 6exzs^3 - 6tyzs^3 - 6eyzs^3 - 2e^3s^3 - 3x^4s^2 - 3y^4s^2 - \\ & 3z^4s^2 - 9e^2t^2s^2 + 3t^2x^2s^2 - 6etx^2s^2 + 3e^2x^2s^2 + 3t^2y^2s^2 - \\ & 3x^2y^2s^2 - 6ety^2s^2 + 3e^2y^2s^2 + 3t^2z^2s^2 - 3x^2z^2s^2 - 3y^2z^2s^2 - \\ & 6etz^2s^2 + 6xy z^2s^2 + 3e^2z^2s^2 + 6t^2xys^2 + 6etxys^2 + 6e^2xys^2 + \\ & 6xy^2zs^2 + 6t^2xzs^2 + 6etxzs^2 + 6e^2xzs^2 + 6t^2yzs^2 + 6x^2yzs^2 + \\ & 6etyzs^2 + 6e^2yzs^2 + 6et^4s + 6txy^3s + 6exy^3s + 6txz^3s + \\ & 6exz^3s + 6tyz^3s + 6eyz^3s - 6et^2x^2s - 6e^2tx^2s - 6et^2y^2s - \\ & 6tx^2y^2s - 6ex^2y^2s - 6e^2ty^2s - 6et^2z^2s - 6tx^2z^2s - 6ex^2z^2s - \\ & 6ty^2z^2s - 6ey^2z^2s - 6e^2tz^2s - 6txyz^2s - 6exyz^2s + 6e^4ts + \\ & 6tx^3ys + 6ex^3ys - 6t^3xys + 6et^2xys + 6e^2txys - 6e^3xys + \\ & 6tx^3zs + 6ex^3zs + 6ty^3zs + 6ey^3zs - 6txy^2zs - 6exy^2zs - \\ & 6t^3xzs + 6et^2xzs + 6e^2txzs - 6e^3xzs - 6t^3yzs + 6et^2yzs - \\ & 6tx^2yzs - 6ex^2yzs + 6e^2tyzs - 6e^3yzs - t^6 + x^6 + y^6 + z^6 - \\ & 3t^2x^4 - 3e^2x^4 - 3t^2y^4 - 3e^2y^4 - 3t^2z^4 - 6xy z^4 - 3e^2z^4 - 2e^3t^3 + \\ & 2x^3y^3 + 6etx y^3 + 2x^3z^3 + 2y^3z^3 + 6etx z^3 + 6ety z^3 + 3t^4x^2 + \\ & 3e^2t^2x^2 + 3e^4x^2 + 3t^4y^2 + 3e^2t^2y^2 - 3t^2x^2y^2 - 6etx^2y^2 - \\ & 3e^2x^2y^2 + 3e^4y^2 + 3t^4z^2 + 3e^2t^2z^2 - 3t^2x^2z^2 - 6etx^2z^2 - \\ & 3e^2x^2z^2 - 3t^2y^2z^2 + 9x^2y^2z^2 - 6ety^2z^2 - 3e^2y^2z^2 + 6t^2xy z^2 - \\ & 6etxy z^2 + 6e^2xy z^2 + 3e^4z^2 + 6etx^3y - 6et^3xy + 6e^2t^2xy - \\ & 6e^3txy - 6xy^4z + 6etx^3z + 6ety^3z + 6t^2xy^2z - 6etxy^2z + \\ & 6e^2xy^2z - 6et^3xz + 6e^2t^2xz - 6e^3txz - 6x^4yz - 6et^3yz + \\ & 6e^2t^2yz + 6t^2x^2yz - 6etx^2yz + 6e^2x^2yz - 6e^3txyz - e^6 \end{aligned}$$

factorize(det[[e,t,s,x,y,z],[t,s,e,z,x,y],[s,e,t,y,z,x],[x,y,z,e,t,s],[y,z,x,s,e,t],[z,x,y,t,s,e]])



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Input interpretation

factor

$$\begin{vmatrix} e & t & s & x & y & z \\ t & s & e & z & x & y \\ s & e & t & y & z & x \\ x & y & z & e & t & s \\ y & z & x & s & e & t \\ z & x & y & t & s & e \end{vmatrix}$$

$$|S_3| = 6 = 1^2 + 1^2 + 2^2.$$

Exercise. Compute the Cayley det.

for $\mathbb{Z}/p\mathbb{Z}$ and expand it as

|m| is the determinant

a product of linear factors.

Result

$$-(s+t-x-y-z+e)(s+t+x+y+z+e) \\ (s^2-st-es+t^2-et-x^2+xy+xz-y^2+yz-z^2+e^2)^2$$

Def. Let G be a group,
let V be a fin. dim'l vector space over $\mathbb{K}$.

A representation of G in V is

a homomorphism $\rho: G \longrightarrow GL(V)$

$\nearrow$
nondeg. linear operators
on V .

$$\rho: G \longrightarrow GL(V)$$

$$g \longmapsto \rho(g): V \longrightarrow V$$

$$\bullet \rho(g_1) \rho(g_2) = \rho(g_1 g_2)$$

$$\bullet \rho(e) = Id_V = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}.$$

$\rho(g)$ are nondeg.!

Examples: 0) Trivial repr.: $G \rightarrow GL(V)$
 $g \mapsto \text{Id}_V$

1) $M \subset \mathbb{R}^n = V$ M bounded subset

$\text{Sym } M \subset GL(V)$: group of $g: g(M) = M$.

$\text{Sym } M$ acts on V :

$g \mapsto$ linear map $V \rightarrow V$.

2) Permutation representation:

$G \curvearrowright X$ finite set

$V_X = \text{Span} \{e_x \mid x \in X\}$

$$g \cdot e_x = e_{g \cdot x}$$

2a) $S_n \curvearrowright \mathbb{C}^n$ $\sigma \cdot e_i = e_{\sigma(i)}$
tautological rep. of S_n .

2b) (Left) Regular representation.

$G \curvearrowright G$ left shifts.

$$\mathbb{C}G = \text{Span} \{e_h \mid h \in G\}.$$

$$g \circ h = g \cdot h$$

$$g \cdot e_h = e_{gh}.$$

2c) right regular rep.

$$G \curvearrowright G$$

~~$$g \circ h = hg$$~~

$$g \circ h = hg^{-1}$$

BAD
def.

$$(g_1 g_2) \circ h = g_1 \circ (g_2 \circ h)$$

$$(g_1 g_2) \circ h$$

$$g_1 \circ (g_2 \circ h)$$

"

"

~~$$h \cdot g_1 g_2 = hg_2 g_1$$~~

$$h(g_1 g_2)^{-1}$$

$$h \cdot g_2^{-1} g_1^{-1}$$

$$(AB)^{-1} = B^{-1}A^{-1}.$$

- W. Fulton, J. Harris
Repr. theory: a First Course
- J.-P. Serre, Reps of Finite Groups
- P. Etingof et al. (also on arXiv)

Terminology : $\rho: G \rightarrow GL(V)$

V - rep. space

or just
representation.

$\dim \rho = \dim V$.

$$\rho: g \mapsto \rho(g) = g_V = g.$$

- Producing new reps.

$V, W \rightsquigarrow V \oplus W$ direct sum

$$\begin{array}{l} \searrow \\ \searrow \\ \searrow \end{array} \begin{array}{l} V \otimes W \\ V^* \\ \text{Hom}(V, W) \cong V^* \otimes W \end{array}$$

$$\rho_1: G \rightarrow GL(V)$$

$$\rho_2: G \rightarrow GL(W)$$

$$\rho_1 \oplus \rho_2: G \rightarrow GL(V \oplus W)$$

	$v_1 \dots v_n$	$w_1 \dots w_m$
v_1 $\vdots$ v_n	$\rho_1(g)$	0
w_1 $\vdots$ w_m	0	$\rho_2(g)$

Matrix of $\rho_1 \oplus \rho_2(g)$ are block diag. simultaneously.

• Subrepresentation:

$$\rho: G \rightarrow GL(V)$$

is a subspace $U \subset V$ s.t. $\rho(g)U \subset U \quad \forall g \in G$

$0, V \subset V$ are always subreps.

Exercise. Define the quotient repr.:

V -rep. V/U
 U -subrep $?$

Def. 1. $\rho: G \rightarrow GL(V)$ is said to be indecomposable
if it cannot be presented as $V = W \oplus U$,
 $W, U \subset V$ subreps, $W, U \neq 0$.

Def. 2. $\rho: G \rightarrow GL(V)$ is said to be irreducible
if V does not have nontrivial subreps.

$$(2) \Rightarrow (1)$$

Is the inverse true? **NO.** (Bad News)

Counterexample: $(\mathbb{Z}, +) \curvearrowright \mathbb{C}^2 = \langle e_1, e_2 \rangle$

$$\rho: n \mapsto \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

$$\rho(n): e_1 \mapsto e_1, e_2 \mapsto e_2 + ne_1.$$

Exercise. Check that this is a rep. (grp homomorphism)

$\text{Span}\{e_1\}$ is a subrep, the only one!

$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ has a unique invariant line $\text{Span}(e_1)$

This rep. is indecomposable, but not irreducible!



semisimplicity.

Good news: For G finite, ① $\Leftrightarrow$ ②.

Any subrep. has a direct complement.

Def. A rep. V of a group G is said to be completely reducible if for $\forall$ subrep $U \subseteq V$ $\exists$ subrep $W \subseteq V$:
 $U \oplus W = V$

Thm. (Maschke). For G finite group,
 V rep. over a field K , $\text{char } K \nmid |G|$,
 V is completely reducible.

Today: $K = \mathbb{R}$ or $\mathbb{C}$.

Idea: if $(\cdot, \cdot)$ - Euclidian or Hermitian
inner product on V ,

then for $U \subset V$ we have $U^\perp \oplus U = V$.

Def. An inner product on V is called

G -invariant, if $(gv, gw) = (v, w) \quad \forall g \in G$.

Lemma. For each real or cplx rep. V of

a finite grp G , V has a G -inv.

Euclidian (resp. Hermitian) inner product.

Pf. Let $(\cdot, \cdot)$ be an arb. Eucl./Herm form.
Define its average over G as follows:

$$(v, w)_G = \frac{1}{|G|} \sum_{g \in G} (gv, gw).$$

Obviously $(\cdot, \cdot)_G$ is G -invariant:

$$\begin{aligned} \text{indeed, } (hv, hw)_G &= \frac{1}{|G|} \sum_{g \in G} (hgv, hgw) = \\ &= (v, w)_G. \end{aligned}$$

Pf of Maschke's theorem:

Let V be a rep. of G , over $\mathbb{R}$ or $\mathbb{C}$.
 $U \subset V$ subrep.

Consider a G -inv. Eucl./Herm. form $(\cdot, \cdot)$.

$$W = U^\perp = \{ w \in V \mid (u, w)_G = 0 \quad \forall u \in U \}.$$

Then $U \oplus W = V$ as vector spaces.

Need to show: W is a subrep:

i.e., $gw \in W$ for any $g \in G$, $w \in W$



$$(gw, u)_G = 0 \quad \forall u \in U$$

"invariance"

$$(g^{-1} \cdot gw, g^{-1}u)_G = (w, \overbrace{g^{-1}u}^u)_G = 0.$$

So $gw \perp u \quad \forall u \in U$, hence $gw \in W$.

W is indeed a subrep. $\square$

Cor. Any real or cplx rep. of a finite group is completely reducible.

Exercise: Find a rep. of G over $\mathbb{F}_p$ that is not completely reducible.